

Curve test for enhanced ind-sheaves and holonomic D -modules

Takuro Mochizuki

Abstract

Recently, the Riemann-Hilbert correspondence was generalized in the context of general holonomic D -modules by A. D'Agnolo and M. Kashiwara. Namely, they proved that their enhanced de Rham functor gives a fully faithfully embedding of the derived category of cohomologically holonomic complexes of D -modules into the derived category of complexes of real constructible enhanced ind-sheaves.

In this paper, we study a condition when a complex of real constructible enhanced ind-sheaves K is induced by a cohomologically holonomic complex of D -modules. We characterize such K in terms of the restriction of K to holomorphic curves.

MSC2010: 14F10, 14F05, 32C38.

1 Introduction

Main result In [3], A. D'Agnolo and M. Kashiwara established the Riemann-Hilbert correspondence for holonomic \mathcal{D} -modules, by generalizing the classical Riemann-Hilbert correspondence between complexes of regular holonomic \mathcal{D} -modules and cohomologically \mathbb{C} -constructible complexes. They introduced the concept of \mathbb{R} -constructible enhanced ind-sheaves on the bases of the theory of ind-sheaves [15, 16]. For any complex manifold X , they constructed the de Rham functor DR_X^E from the derived category of cohomologically holonomic complexes of \mathcal{D}_X -modules $D_{\mathrm{hol}}^b(\mathcal{D}_X)$ to the derived category of \mathbb{R} -constructible enhanced ind-sheaves $E_{\mathbb{R}-c}^b(IC_X)$, and they proved that DR_X^E is fully faithful and compatible with the 6-operations and the duality. They also gave the reconstruction of a holonomic complex \mathcal{M}^\bullet from its solution complex $\mathrm{Sol}^E(\mathcal{M}^\bullet) \in E_{\mathbb{R}-c}^b(IC_X)$. They efficiently used the study of formal structure and the asymptotic analysis of meromorphic flat bundles ([18], [19], [21], [29], [30], [37]). In [4], they also introduced the natural perversity condition for $E_{\mathbb{R}-c}^b(IC_X)$, and they proved that DR_X^E is exact with respect to the natural t -structure of $D_{\mathrm{hol}}^b(\mathcal{D}_X)$ and the t -structure of $E_{\mathbb{R}-c}^b(IC_X)$ with respect to the perversity condition.

We still have an interesting question. Let $E_{\mathcal{D}}^b(IC_X)$ denote the essential image of DR_X^E . It is natural to ask a condition for an object $K \in E_{\mathbb{R}-c}^b(IC_X)$ to be contained in $E_{\mathcal{D}}^b(IC_X)$. In the regular singular case, it is given by the cohomological \mathbb{C} -constructibility condition. As far as the author knows, such a clear condition has not yet been given in the enhanced case.

In this paper, we study “a curve test”. We consider the full subcategory $E_{\Delta}^b(IC_X) \subset E_{\mathbb{R}-c}^b(IC_X)$ determined by the following condition for objects $K \in E_{\mathbb{R}-c}^b(IC_X)$.

- Set $\Delta := \{|z| < 1\}$. Let $\varphi : \Delta \rightarrow X$ be any holomorphic map. Then, $E\varphi^{-1}(K) \in E_{\mathcal{D}}^b(\Delta)$.

By the compatibility of the de Rham functors DR^E and 6-operations, we clearly have $E_{\mathcal{D}}^b(IC_X) \subset E_{\Delta}^b(IC_X)$. The following is the main theorem of this paper.

Theorem 1.1 (Theorem 10.1) $E_{\Delta}^b(IC_X)$ is equal to $E_{\mathcal{D}}^b(IC_X)$.

Meromorphic flat connections and enhanced ind-sheaves A holonomic \mathcal{D} -module can be locally described as the gluing of meromorphic flat connections given on subvarieties. Hence, it is a key step to study such a characterization for meromorphic flat connections (Theorem 9.3).

Let X be an n -dimensional complex manifold. Let H be a normal crossing hypersurface of X . Let $\mathbf{X}(H)$ denote the bordered space $(X \setminus H, X)$ in the sense of [3, 4]. Suppose that an object $K \in E_{\mathbb{R}-c}^b(IC_{\mathbf{X}(H)})$ satisfies the following condition.

- $K|_{X \setminus H}$ comes from a local system.
- Let $\varphi : \Delta \rightarrow X$ be any holomorphic map such that $\varphi(\Delta \setminus \{0\}) \subset X \setminus H$. Then, $E\varphi^{-1}(K)$ comes from a meromorphic flat bundle on $(\Delta, 0)$.

We would like to show that there exists a meromorphic flat connection (V, ∇) on (X, H) with an isomorphism $\mathrm{DR}_{X(H)}^E(V)[-n] \simeq K$ in $E_{\mathbb{R}-c}^b(IC_{X(H)})$. Once we have a meromorphic flat connection (V, ∇) on (X, H) such that $E\varphi^{-1}(K) \simeq \mathrm{DR}_{X(H)}^E \varphi^*(V, \nabla)[-1]$ in a natural way for any $\varphi : \Delta \rightarrow X$ as above, then it is not so difficult to prove that $K \simeq \mathrm{DR}_{X(H)}^E(V, \nabla)[-n]$ (Proposition 3.31). So, we would like to construct such V .

We explain a brief outline for the construction of such V in the essential case $n = 2$. Suppose that we are given a good set of ramified irregular values \mathcal{I}_P at a smooth point $P \in H$ and a multiplicity function $\mathbf{m}_P : \mathcal{I}_P \rightarrow \mathbb{Z}_{\geq 0}$ such that the following holds.

(GA) For any holomorphic map $\varphi : \Delta \rightarrow X$ such that $\varphi(\Delta \setminus \{0\}) \subset X \setminus H$ and $\varphi(0)$ is close to P , we have $\mathrm{Irr}(E\varphi^{-1}K) \simeq \varphi^*\mathcal{I}_P$ compatible with the multiplicity.

Then, we can construct a good meromorphic flat bundle (V, ∇) on a neighbourhood X_P of P in X , with an isomorphism $\mathrm{DR}_{X_P(H \cap X_P)}^E(V)[-2] \simeq K$ (Proposition 3.32). We have a similar condition at cross points of H . Although the condition **(GA)** is not always satisfied even if K comes from a meromorphic flat connection, it is our strategy to modify X in a birational way so that the condition **(GA)** is satisfied.

We have a stratification $X \setminus H = \coprod \mathcal{C}$ by locally closed subanalytic subsets such that we have subanalytic functions $h_i^{\mathcal{C}}$ on (\mathcal{C}, X) such that $\pi^{-1}(\mathcal{C}) \otimes K \simeq \bigoplus \mathbb{C}^E \otimes \mathbb{C}_{t \geq h_i^{\mathcal{C}}}$. By using such a local description of K , we can show that we have a 0-dimensional closed subanalytic subset $Z \subset H$ such that any $P \in H \setminus Z$ is smooth point of H , and that **(GA)** is satisfied for K at P . As a result, we obtain a meromorphic flat bundle (V, ∇) on $(X \setminus Z, H \setminus Z)$ with an isomorphism $\mathrm{DR}^E(V)[-2] \simeq K|_{X \setminus Z}$ (Proposition 4.13).

We take the complex blowing up of $\psi_1 : X_1 \rightarrow X$ at the points of Z , and we set $H_1 := \psi_1^{-1}(H)$. We have finite points $Z_1 \subset H_1$ such that we have a meromorphic flat bundle (V_1, ∇) on $(X_1 \setminus Z_1, H_1 \setminus Z_1)$ with an isomorphism $\mathrm{DR}^E(V_1, \nabla)[-2] \simeq E\psi_1^{-1}(K)|_{X_1 \setminus Z_1}$. Then, we take again the complex blowing up $\psi_2 : X_2 \rightarrow X_1$ at Z_1 . By continuing the procedure successively, we obtain the following sequence:

$$\cdots \rightarrow (X_\ell, H_\ell) \xrightarrow{\psi_\ell} (X_{\ell-1}, H_{\ell-1}) \xrightarrow{\psi_{\ell-1}} \cdots \rightarrow (X_1, H_1) \xrightarrow{\psi_1} (X, H)$$

Here, we have finite points $Z_j \subset H_j$ such that ψ_{j+1} is the complex blowing up at Z_j and that $Z_{j+1} \subset (\psi_{j+1})^{-1}(Z_j)$. We would like to prove that this will stop after finite steps, i.e., we can take Z_ℓ as the empty set if ℓ is large. It is argued in §6 for the composition of complex blowings up at cross points, and in §9.1 for the mixture of complex blowings up at cross points and complex blowings up at smooth points, after the preliminary in §7–§8. This type of issue appeared in the study of Sabbah's conjecture, i.e., the higher dimensional generalization of the Hukuhara-Levelt-Turrittin theorem.

Complex blowing up and real blowing up For this problem, we shall study the pull back of subanalytic functions defined on locally closed subanalytic subsets by the composition of ψ_p . (See [2] and [9] for the general theory of subanalytic sets.) Let \mathcal{C} be a locally closed subanalytic subset in X , and let f be a subanalytic function on (\mathcal{C}, X) . Let $\varpi_j : \tilde{X}_j(H_j) \rightarrow X_j$ be the oriented blowing up of X_j along H_j . Let Q_j be a point of $\varpi_j^{-1}(H_j^{(1)})$. We take a small neighbourhood \mathcal{U}_{Q_j} of Q_j in $\tilde{X}_j(H_j)$. Let $\Phi_j : \tilde{X}_j(H_j) \rightarrow X$ be the morphism induced by the composition of ψ_p . If $\Phi_j(\mathcal{U}_{Q_j}) \cap \mathcal{C} \neq \emptyset$, we are lead to study the function $\Phi_j^*(f)$ on $\Phi_j^{-1}(\mathcal{C}) \cap \mathcal{U}_{Q_j}$, and to show that $\Phi_j^*(f)$ is “simpler” than the original f .

It is not easy to deal with f and $\Phi_j^*(f)$ directly. According to [35], we have a rectilinearization of f , i.e., we have a locally finite family of real analytic maps $\phi_\alpha : W_\alpha \rightarrow X$ ($\alpha \in \Lambda$) such that each $\phi_\alpha^*(f)$ has a clean description. Here, the morphisms ϕ_α are described as the composition of a finite sequence of local real blowings up. Then, we are naturally lead to the problem to ask whether we have $\alpha_0 \in \Lambda$ and a real analytic map $\Psi_j : \mathcal{U}_{Q_j} \rightarrow W_{\alpha_0}$ such that $\phi_{\alpha_0} \circ \Psi_j$ is equal to the restriction of Φ_j to \mathcal{U}_{Q_j} . Once we have such a lift, we have only to study the pull back of $\phi_{\alpha_0}^*(f)$ by the analytic map Ψ_j^* . Such a lifting problem is studied in §7–§8.

Acknowledgement I am grateful to Masaki Kashiwara for asking this question and for many discussions. This study grew from my effort to understand the interesting works of Andrea D'Agnolo, Kashiwara and Pierre Schapira on enhanced ind-sheaves and the generalized Riemann-Hilbert correspondence. I also thank Giovanni Morando who first attracted my attention to the theory of ind-sheaves. I thank Claude Sabbah for his kindness and for discussions on many occasions. I am grateful to Akira Ishii and Yoshifumi Tsuchimoto for their constant encouragement.

This work was partially supported by the Grant-in-Aid for Scientific Research (S) (No. 24224001), the Grant-in-Aid for Scientific Research (S) (No. 16H06335), and the Grant-in-Aid for Scientific Research (C) (No. 15K04843), Japan Society for the Promotion of Science.

2 Preliminary

2.1 Rectilinearization

2.1.1 Rectilinearization of subanalytic subsets

A subset $A \subset \mathbb{R}^n$ is called a quadrant if we have a decomposition $\{1, \dots, n\} = I_0 \sqcup I_+ \sqcup I_-$ for which

$$A = \{(x_1, \dots, x_n) \mid x_i = 0 \ (i \in I_0), \ x_i > 0 \ (i \in I_+), \ x_i < 0 \ (i \in I_-)\}.$$

Let B be any subanalytic set in an n -dimensional real analytic manifold X . A rectilinearization of B is a locally finite family of real analytic maps $\phi_\alpha : W_\alpha \rightarrow X$ ($\alpha \in \Lambda$) with the following property.

- Each W_α is equipped with a real analytic coordinate system (x_1, \dots, x_n) with which we have $W_\alpha \simeq \mathbb{R}^n$.
- Each ϕ_α is the composition of a finite sequence of local real blowings up with smooth real analytic centers. Namely, we have a factorization of ϕ_α :

$$W_\alpha = W_\alpha^{(k(\alpha))} \xrightarrow{\phi_\alpha^{(k(\alpha))}} W_\alpha^{(k(\alpha)-1)} \xrightarrow{\phi_\alpha^{(k(\alpha)-1)}} \dots \xrightarrow{\phi_\alpha^{(2)}} W_\alpha^{(1)} \xrightarrow{\phi_\alpha^{(1)}} W_\alpha^{(0)} = X \quad (1)$$

Moreover, we have subanalytic open subsets $U_\alpha^{(\ell)} \subset W_\alpha^{(\ell)}$ and a closed real analytic submanifold $C_\alpha^{(\ell)} \subset U_\alpha^{(\ell)}$ so that $\phi_\alpha^{(\ell+1)}$ is the real blowing up of $U_\alpha^{(\ell)}$ along $C_\alpha^{(\ell)}$. We also impose that $C_\alpha^{(\ell)}$ is subanalytic in $W_\alpha^{(\ell)}$. Note that $C_\alpha^{(\ell)}$ can be empty.

- We have compact subsets $K_\alpha \subset W_\alpha$ such that $\bigcup_{\alpha \in \Lambda} \phi_\alpha(K_\alpha) = X$.
- For each α , $\phi_\alpha^{-1}(B)$ is rectilinearized, i.e., $\phi_\alpha^{-1}(B)$ is a union of quadrants with respect to the coordinate system.

Let us recall the following fundamental theorem due to Hironaka [9]. (See also [2] and [35].)

Proposition 2.1 *For any subanalytic subset B in X , we have a rectilinearization.* ■

2.1.2 Ramified normal crossing functions on quadrants

Recall that an analytic function g on \mathbb{R}^n is called normal crossing if $g = \prod_{i=1}^n x_i^{m_i} \times g_0$, where m_i are non-negative integers and g_0 is a nowhere vanishing analytic function on \mathbb{R}^n .

We have a variant of the concept for functions on quadrants. Let Q be any n -dimensional quadrant in \mathbb{R}^n . An analytic function f on Q is called normal crossing if $f = \prod_{i=1}^n x_i^{m_i} \times g$, where g is a nowhere vanishing analytic function on a neighbourhood of the closure \overline{Q} , and $(m_1, \dots, m_n) \in \mathbb{Z}_{\geq 0}^n$.

Let $Q_0 \subset \mathbb{R}^n$ denote the n -dimensional quadrant given by $\{x_i > 0 \ (i = 1, \dots, n)\}$. Let $\{1, \dots, n\} = I_+ \sqcup I_-$ be a decomposition. Let $Q(I_+, I_-)$ be the n -dimensional quadrant in \mathbb{R}^n given by $Q(I_+, I_-) = \{x_i > 0 \ (i \in I_+), \ x_i < 0 \ (i \in I_-)\}$. For any tuple of positive integers (ρ_1, \dots, ρ_n) , we have the homeomorphism $F : \overline{Q_0} \rightarrow \overline{Q(I_+, I_-)}$ given by $F(y_1, \dots, y_n) = (\epsilon_1 y^{\rho_1}, \dots, \epsilon_n y^{\rho_n})$, where $\epsilon_i = 1 \ (i \in I_+)$ and $\epsilon_i = -1 \ (i \in I_-)$. Such homeomorphisms are called ramified analytic isomorphism of $\overline{Q_0}$ and $\overline{Q(I_+, I_-)}$.

An analytic function f on an n -dimensional quadrant Q is called ramified normal crossing if we have a ramified isomorphism $F : \overline{Q_0} \rightarrow \overline{Q}$ such that $F^*(f)$ is normal crossing. We have the continuous extension \bar{f} of f on \overline{Q} , and the restrictions $\bar{f}|_{Q'}$ are ramified normal crossing analytic functions on Q' for any quadrant contained in \overline{Q} , where we regard Q' as a quadrant in $\mathbb{R}^{\dim Q'}$.

Let $\text{RNC}_+(Q)$ denote the set of ramified normal crossing functions on Q . We set $\text{RNC}_-(Q) := \{1/f \mid f \in \text{RNC}_+(Q)\}$ and $\text{RNC}(Q) := \text{RNC}_+(Q) \cup \text{RNC}_-(Q) \sqcup \{0\}$.

Let $\mathbb{R}_{\mathbf{y}}^r = \{(y_1, \dots, y_r) \in \mathbb{R}^r\}$ and $\mathbb{R}_{\mathbf{x}}^n = \{(x_1, \dots, x_n) \in \mathbb{R}^n\}$.

Lemma 2.2 *Let $\phi : \mathbb{R}_{\mathbf{y}}^r \rightarrow \mathbb{R}_{\mathbf{x}}^n$ be a real analytic map such that each $\phi^*(x_i)$ ($i = 1, \dots, n$) is normal crossing or constantly 0.*

- *Let Q_1 be a quadrant of $\mathbb{R}_{\mathbf{x}}^n$. Then, $\phi^{-1}(Q_1)$ is the union of the quadrants Q of $\mathbb{R}_{\mathbf{y}}^r$ such that $Q \subset \phi^{-1}(Q_1)$.*
- *Let Q_1 be a quadrant of $\mathbb{R}_{\mathbf{x}}^n$. Let $f \in \text{RNC}(Q_1)$, where we naturally regard Q_1 as a quadrant in $\mathbb{R}^{\dim Q_1}$. Let Q_2 be any quadrant of $\mathbb{R}_{\mathbf{y}}^r$ such that $\phi(Q_2) \subset Q_1$. Then, we have $\phi^*(f) \in \text{RNC}(Q_2)$.*

Proof By the assumption, one of the following holds for each $i = 1, \dots, n$; (i) we have $\phi^*(x_i) = 0$, (ii) we have $\phi^*(x_i) = a_i \cdot \prod_{k=1}^r y_k^{m(i)_k}$, where a_i is nowhere vanishing on $\mathbb{R}_{\mathbf{y}}^r$, and $m(i)_k \in \mathbb{Z}_{\geq 0}$.

For any quadrant Q in $\mathbb{R}_{\mathbf{y}}^r$, one of the following holds; (i) $\prod_{k=1}^r y_k^{m(i)_k}$ are constantly 0 on Q , (ii) $\prod_{k=1}^r y_k^{m(i)_k}$ is positive on Q , (iii) $\prod_{k=1}^r y_k^{m(i)_k}$ is negative on Q . Hence, one of the following holds; (i) $\phi(Q) \subset Q_1$, (ii) $\phi(Q) \cap Q_1 = \emptyset$. Then, we obtain the first claim.

Let us study the second claim. It is enough to consider the case $f \in \text{RNC}_+(Q_1)$. We may assume that Q_1 is given as $\bigcap_{i=1}^{\ell} \{x_i > 0\} \cap \bigcap_{i=\ell+1}^n \{x_i = 0\}$, and that Q_2 is given as $\bigcap_{k=1}^p \{y_k > 0\} \cap \bigcap_{k=p+1}^r \{y_k = 0\}$. For any $i = 1, \dots, \ell$, because $\phi^*(x_i) > 0$ on Q_2 , we have $a_i > 0$ on Q_2 and $m(i)_k = 0$ for $k = p+1, \dots, r$.

We naturally regard $\mathbb{R}_{\mathbf{x}}^{\ell} = \{(x_1, \dots, x_{\ell})\}$ as a subspace of $\mathbb{R}_{\mathbf{x}}^n$. Then, Q_1 is a quadrant in $\mathbb{R}_{\mathbf{x}}^{\ell}$. Similarly, we naturally regard $\mathbb{R}_{\mathbf{y}}^p = \{(y_1, \dots, y_p)\}$ as a subspace of $\mathbb{R}_{\mathbf{y}}^r$, and Q_2 is a quadrant of $\mathbb{R}_{\mathbf{y}}^p$.

We have a positive integer ρ with the following property.

- Let $\Phi : \mathbb{R}_{\mathbf{x}}^{\ell} \rightarrow \mathbb{R}_{\mathbf{x}}^{\ell}$ be given by $\Phi(x_1, \dots, x_{\ell}) = (x_1^{\rho}, \dots, x_{\ell}^{\rho})$. Let $\Phi_{Q_1} : Q_1 \rightarrow Q_1$ denote the induced map. Then, $\Phi_{Q_1}^*(f)$ is a real analytic normal crossing function on Q_1 .

For $i = 1, \dots, \ell$, we have $\phi^*(x_i^{1/\rho})|_{\mathbb{R}_{\mathbf{y}}^p} = a_i^{1/\rho} \prod_{k=1}^p y_k^{m(i)_k/\rho}$. Let $\Psi : \mathbb{R}_{\mathbf{y}}^p \rightarrow \mathbb{R}_{\mathbf{y}}^p$ be given by $\Psi(y_1, \dots, y_p) = (y_1^{\rho}, \dots, y_p^{\rho})$. Then, $\Psi^* \phi^*(x_i^{1/\rho})$ are the restriction of normal crossing real analytic functions on $\mathbb{R}_{\mathbf{y}}^p$. Hence, we have $F : \mathbb{R}_{\mathbf{y}}^p \rightarrow \mathbb{R}_{\mathbf{x}}^{\ell}$ such that $F(Q_2) \subset Q_1$, and $\Phi \circ F = \phi \circ \Psi$. We also have $F^*(x_i)$ ($i = 1, \dots, \ell$) are normal crossing. Let Ψ_{Q_2} and F_{Q_2} denote the restriction of Ψ and F to Q_2 , respectively. Then, $F_{Q_2}^* \Phi_{Q_1}^*(f|_{Q_1}) = \Psi_{Q_2}^* \phi^*(f|_{Q_1})$ is a real analytic normal crossing function on Q_2 . Thus, we obtain the second claim. \blacksquare

We have the following consequence of [2, Theorem 4.4].

Lemma 2.3 *Let Q_1 be any quadrant of \mathbb{R}^n . Suppose that we are given $f \in \text{RNC}(Q_1)$ and analytic functions g_1, \dots, g_m on \mathbb{R}^n such that each g_i is not constantly 0. Then, we have a rectilinearization $\{(W_{\alpha}, \phi_{\alpha}) \mid \alpha \in \Lambda\}$ of Q_1 with the following property.*

- $\phi_{\alpha}^*(f)|_Q \in \text{RNC}(Q)$ for any quadrant Q contained in $\phi_{\alpha}^{-1}(Q_1)$.
- $\phi_{\alpha}^*(g_j)$ are normal crossing functions on W_{α} .

Proof Set $h := \prod_{i=1}^n x_i \times \prod_{j=1}^m g_j$. By [2, Theorem 4.4], we have a locally finite family of real analytic maps $\phi_{\alpha} : W_{\alpha} \rightarrow \mathbb{R}^n$ ($\alpha \in \Lambda$) with the following property.

- Each ϕ_{α} is the composition of a finite sequence of local real blowings up.
- Each W_{α} is equipped with a coordinate (y_1, \dots, y_n) , for which we have $W_{\alpha} \simeq \mathbb{R}^n$, and $\phi_{\alpha}^{-1}(h)$ is normal crossing on W_{α} .

Because $\phi_\alpha^{-1}(h)$ is normal crossing, we obtain that $\phi_\alpha^*(x_i)$ and $\phi_\alpha^*(g_j)$ are normal crossing on W_α . By Lemma 2.2, we obtain that $\{(W_\alpha, \phi_\alpha) \mid \alpha \in \Lambda\}$ is a rectilinearization of any quadrant Q_1 of \mathbb{R}^n . We also obtain from Lemma 2.2 that $\phi_\alpha^*(f)|_Q \in \text{RNC}(Q)$ for any quadrant $Q \subset \phi_\alpha^{-1}(Q_1)$. \blacksquare

We give a remark on rectilinearization of subanalytic subsets.

Lemma 2.4 *Let B_1, \dots, B_m be subanalytic subsets in a real analytic manifold X . We have a locally finite family of real analytic maps $\{(W_\alpha, \phi_\alpha) \mid \alpha \in \Lambda\}$ which is a rectilinearization of each B_i .*

Proof We use an induction on m . Suppose that we already have a locally finite family of analytic maps $\{(W_\alpha, \phi_\alpha) \mid \alpha \in \Lambda\}$ which is a rectilinearization of B_i ($i = 1, \dots, m-1$). We can take subanalytic compact subsets $K_\alpha \subset W_\alpha$ such that $\bigcup_\alpha \phi_\alpha(K_\alpha) = X$. By Proposition 2.1, we have rectilinearizations $\{(V_{\alpha,\beta}, \psi_{\alpha,\beta}) \mid \beta \in \Gamma(\alpha)\}$ of $K_\alpha \cap \phi_\alpha^{-1}(B_m)$. Each W_α is equipped with the coordinate $(x_1^\alpha, \dots, x_n^\alpha)$. Each $V_{\alpha,\beta}$ is equipped with the coordinate $(y_1^{\alpha,\beta}, \dots, y_n^{\alpha,\beta})$. We set $h_{\alpha,\beta} := \prod_{j=1}^n y_j^{\alpha,\beta} \cdot \prod_{i=1}^n \psi_{\alpha,\beta}^* x_i^\alpha$. We take subanalytic compact subsets $L_{\alpha,\beta} \subset V_{\alpha,\beta}$ such that $\bigcup_\beta \psi_{\alpha,\beta}(L_{\alpha,\beta})$ contains a neighbourhood of K_α . Applying [2, Theorem 4.4] to $h_{\alpha,\beta}$, we also have a locally finite family of real analytic maps $\lambda_{\alpha,\beta,\gamma} : Y_{\alpha,\beta,\gamma} \rightarrow V_{\alpha,\beta}$ ($\lambda_{\alpha,\beta,\gamma} \in \Upsilon(\alpha, \beta)$) such that (i) each $\lambda_{\alpha,\beta,\gamma}$ is the composition of a finite sequence of local real blowings up, (ii) $\lambda_{\alpha,\beta,\gamma}^* h_{\alpha,\beta}$ are normal crossing. Then, as in the proof of Lemma 2.3, we can observe that each $\lambda_{\alpha,\beta,\gamma}^{-1}(B_j)$ is the union of quadrants contained in $\lambda_{\alpha,\beta,\gamma}^{-1}(B_j)$.

For each $\alpha \in \Lambda$, we have the finite subset

$$\Gamma_1(\alpha) := \{\beta \in \Gamma(\alpha) \mid \psi_{\alpha,\beta}(V_{\alpha,\beta}) \cap K_\alpha \neq \emptyset\}.$$

For each $\beta \in \Gamma_1(\alpha)$, we have the finite subset

$$\Upsilon_1(\alpha, \beta) := \{\gamma \in \Upsilon(\alpha, \beta) \mid \lambda_{\alpha,\beta,\gamma}(Y_{\alpha,\beta,\gamma}) \cap L_{\alpha,\beta} \neq \emptyset\}.$$

Let $\tilde{\Lambda}$ denote the set of (α, β, γ) , where $\alpha \in \Lambda$, $\beta \in \Gamma_1(\alpha)$ and $\gamma \in \Upsilon_1(\alpha, \beta)$. For each $(\alpha, \beta, \gamma) \in \tilde{\Lambda}$, we set $\tilde{\phi}_{\alpha,\beta,\gamma} := \phi_\alpha \circ \psi_{\alpha,\beta} \circ \lambda_{\alpha,\beta,\gamma}$. Then, the locally finite family of real analytic maps $(Y_{\alpha,\beta,\gamma}, \tilde{\phi}_{\alpha,\beta,\gamma})$ has the desired property. \blacksquare

Similarly, we can obtain the following refinement.

Lemma 2.5 *Let B_i ($i \in S$) be a locally finite family of subanalytic subsets in a real analytic manifold X . We have a locally finite family of real analytic maps $\{(W_\alpha, \phi_\alpha) \mid \alpha \in \Lambda\}$ which is a rectilinearization of each B_i ($i \in S$).* \blacksquare

2.1.3 Rectilinearization of subanalytic functions

Let B be any subanalytic set in a real analytic manifold X . A function $f : B \rightarrow \mathbb{R}$ is called subanalytic on (B, X) if the graph of f is a subanalytic subset of $X \times \mathbb{P}^1(\mathbb{R})$. The following is the rectilinearization theorem for subanalytic functions due to Parusiński [35, Theorem 2.7]. (See also [20, Theorem 3.4].)

Proposition 2.6 *Let U be an open subanalytic subset of \mathbb{R}^n , and let $f : U \rightarrow \mathbb{R}$ be a function which is subanalytic on (U, \mathbb{R}^n) and continuous on U . Then, there exists a rectilinearization $\{(W_\alpha, \phi_\alpha) \mid \alpha \in \Lambda\}$ of U with the following property.*

- Let Q be any n -dimensional quadrant contained in $\phi_\alpha^{-1}(U)$. Then, $\phi_\alpha^*(f)|_Q \in \text{RNC}(Q)$. \blacksquare

For any subanalytic subset $U \subset \mathbb{R}^n$, let $\text{Quad}_n(U, \mathbb{R}^n)$ denote the set of the n -dimensional quadrants contained in U .

Corollary 2.7 *Let U be a subanalytic open set in \mathbb{R}^n such that U is equal to the union of the quadrants contained in U . Let f_1, \dots, f_ℓ be subanalytic functions on (U, \mathbb{R}^n) such that (i) f_i are continuous on U , (ii) $f_i|_Q \in \text{RNC}(Q)$ for each $Q \in \text{Quad}_n(U, \mathbb{R}^n)$. Let h_1, \dots, h_p be analytic functions on \mathbb{R}^n . Let g_1, \dots, g_m be subanalytic functions on U . Then, we have a rectilinearization $\{(W_\alpha, \phi_\alpha) \mid \alpha \in \Lambda\}$ for U such that the following holds:*

- $\phi_\alpha^*(f_i)|_Q, \phi_\alpha^*(g_j)|_Q \in \text{RNC}(Q)$ for each $Q \in \text{Quad}_n(\phi_\alpha^{-1}(U), W_\alpha)$, where we regard $W_\alpha \simeq \mathbb{R}^n$ by the coordinate system.
- Each $\phi_\alpha^*(h_j)$ is normal crossing on $W_\alpha \simeq \mathbb{R}^n$, or constantly 0.

Proof Let us study the case $m = 1$. Let (x_1, \dots, x_n) be the global coordinate of \mathbb{R}^n . By Proposition 2.6 and Lemma 2.3, we have a rectilinearization $\{(W_\alpha, \phi_\alpha) \mid \alpha \in \Lambda\}$ for U such that (i) $\phi_\alpha^*(x_i)$ and $\phi_\alpha^*(h_j)$ are normal crossing on W_α , (ii) $\phi_\alpha^*(g_1) \in \text{RNC}(Q)$ for each $Q \in \text{Quad}_n(\phi_\alpha^{-1}(U), W_\alpha)$. We also obtain that $\phi_\alpha^*(f_i)|_Q \in \text{RNC}(Q)$ from Lemma 2.3. Thus, we are done in the case $m = 1$. We can prove the claim for general m by an easy induction. \blacksquare

Corollary 2.8 *Let U be a subanalytic open subset in \mathbb{R}^n . Let g_1, \dots, g_N be subanalytic functions on (U, \mathbb{R}^n) . Let h_1, \dots, h_ℓ be analytic functions on \mathbb{R}^n . Then, we have a rectilinearization $\{(W_\alpha, \phi_\alpha) \mid \alpha \in \Lambda\}$ of U such that $\phi_\alpha^*(g_i)|_Q \in \text{RNC}(Q)$ for any $Q \in \text{Quad}_n(\phi_\alpha^{-1}(U), W_\alpha)$, and that each $\phi_\alpha^*(h_j)$ is normal crossing on W_α or constantly 0.* \blacksquare

2.1.4 Dominant subharmonic functions

Let H be a closed subanalytic subset of a real analytic manifold M .

Lemma 2.9 *We have a continuous subharmonic function χ_H on M such that $\chi_H^{-1}(0) = H$.*

Proof We take a locally finite covering of M by coordinate neighbourhoods $(\mathcal{U}_\lambda, x_1^\lambda, \dots, x_n^\lambda)$ ($\lambda \in \Lambda$). We have the distance $d_{\mathcal{U}_\lambda}$ on \mathcal{U}_λ induced by the coordinate $(x_1^\lambda, \dots, x_n^\lambda)$ and the standard Euclidean distance on \mathbb{R}^n .

We take a refined locally finite covering \mathcal{V}_μ ($\mu \in \Gamma$) of M such that (i) for each $\mu \in \Gamma$ we have $\lambda(\mu) \in \Lambda$ such that \mathcal{V}_μ is a relatively compact subanalytic subset in $\mathcal{U}_{\lambda(\mu)}$. Let C_μ be the subanalytic subset of $\mathcal{U}_{\lambda(\mu)}$ given as the union of $H \cap \mathcal{U}_{\lambda(\mu)}$ and $\mathcal{U}_{\lambda(\mu)} \setminus \mathcal{V}_\mu$. Let $\chi_\mu : \mathcal{U}_{\lambda(\mu)} \rightarrow \mathbb{R}_{\geq 0}$ be given by

$$\chi_\mu(P) := d_{\mathcal{U}_{\lambda(\mu)}}(P, C_\mu).$$

It is a continuous subanalytic function on $\mathcal{U}_{\lambda(\mu)}$. The support is contained in \mathcal{V}_μ . We extend it to a continuous subanalytic function on M by setting $\chi_\mu(P) = 0$ for any $P \notin \mathcal{U}_{\lambda(\mu)}$. We set $\chi_H(P) := \max_{\mu \in \Gamma} \chi_\mu(P)$. Then, it has the desired property. \blacksquare

We fix any continuous subharmonic function χ_H on M such that $\chi_H^{-1}(0) = H$. Note that χ_H^{-1} is a subanalytic function on $(M \setminus H, M)$.

Lemma 2.10 *Let U be a relatively compact subanalytic open subset of M such that $U \cap H = \emptyset$. Let f be a continuous subanalytic function on U with the following property.*

- For any relatively compact subset $V \subset M \setminus H$, $|f|_{V \cap U}$ is bounded.

Then, we have positive constants N and C such that $|f| \leq C(\chi_H^{-N})|_U$ on U .

Proof We have a rectilinearization $\{(W_\alpha, \phi_\alpha) \mid \alpha \in \Lambda\}$ for f and χ_H . It is easy to see that for each $\alpha \in \Lambda$ we have C_α and N_α such that $|\phi_\alpha^*(f)| \leq C_\alpha \phi_\alpha^*(\chi_H)^{-N_\alpha}$. By using the relative compactness of U , we obtain the claim of the lemma. \blacksquare

2.2 Some singular sets

2.2.1 Singularity of fibrations

Let M be an analytic manifold. Let Y be a relatively compact subanalytic subset in $M \times \mathbb{R}^n$ with $\dim Y = k$. Let Y_k^{sm} denote the set of the k -dimensional smooth points of Y , which is a subanalytic open subset in Y . (See [2, Theorem 7.2].) Let $\overline{Y}_k^{\text{sm}}$ denote the closure of Y_k^{sm} in Y . Let $Y' := Y \setminus Y_k^{\text{sm}}$. We have $\dim Y' \leq k - 1$. The singular locus of $\overline{Y}_k^{\text{sm}}$ is contained in Y' .

Let $\phi : M \times \mathbb{R}^n \rightarrow \mathbb{R}^n$ be the projection. Let $\ell := \dim \phi(Y)$.

Lemma 2.11 *We have closed subanalytic subsets $G \subset \overline{Y}_k^{\text{sm}}$ and $W \subset \phi(Y)$ with the following property.*

- *We have $\dim W < \ell$ and $\dim G < k$.*
- *The singular locus of $\phi(Y)$ is contained in W .*
- *The induced map $Y_k^{\text{sm}} \setminus (G \cup \phi^{-1}(W)) \rightarrow \phi(Y) \setminus W$ is submersive.*
- *For any point $P \in \phi(Y) \setminus W$, we have $\dim((G \cup Y') \cap \phi^{-1}(P)) \leq k - 1 - \ell$.*

In particular, for any point $P \in \phi(Y) \setminus W$, we have $\dim(Y \cap \phi^{-1}(P)) \leq k - \ell$.

Proof We use an induction on $\dim Y$. The claim is clear in the case $\dim Y = 0$. We assume that we have already known the claim in the case $\dim Y < k$.

Let \tilde{Y}_k denote the closure of Y_k^{sm} in $M \times \mathbb{R}^n$, which is compact. We have a k -dimensional real analytic compact manifold \tilde{X}_k with a real analytic map $\tilde{\rho} : \tilde{X}_k \rightarrow M \times \mathbb{R}^n$ such that $\tilde{\rho}(\tilde{X}_k) = \tilde{Y}_k$. We set $X_k := \tilde{\rho}^{-1}(\overline{Y}_k^{\text{sm}})$, and $\rho := \rho|_{X_k}$. We have a closed subset $G \subset \overline{Y}_k^{\text{sm}}$ such that the singular locus of $\overline{Y}_k^{\text{sm}}$ is contained in G and that the induced map $\rho^{-1}(\overline{Y}_k^{\text{sm}} \setminus G) \rightarrow \overline{Y}_k^{\text{sm}} \setminus G$ is a local diffeomorphism.

We set $\tilde{\phi} := \phi \circ \rho : X_k \rightarrow \mathbb{R}^n$. We have $\dim \tilde{\phi}(X_k) = \dim \phi(\overline{Y}_k^{\text{sm}})$. Suppose $\dim \tilde{\phi}(X_k) = \ell$. We have the closed subset H_1 of the points $x \in X_k$ such that $\text{rank}(T_x \tilde{\phi}) < \ell$. Let $\text{Sing } \phi(Y)$ denote the singular locus of $\phi(Y)$. Let C denote the set of the critical values of $X_k \setminus \tilde{\phi}^{-1} \text{Sing } \phi(Y) \rightarrow \phi(Y) \setminus \text{Sing } \phi(Y)$ which is equal to $\tilde{\phi}(H_1) \setminus \text{Sing } \phi(Y)$. It is subanalytic and its measure is 0, and hence $\dim(\tilde{\phi}(H_1) \setminus \text{Sing } \phi(Y)) \leq \ell - 1$. Let W_1 denote the closure of $\tilde{\phi}(H_1) \cup \text{Sing } \phi(Y)$ in $\phi(Y)$. Then, $X_k \setminus \tilde{\phi}^{-1}(W_1) \rightarrow \phi(Y) \setminus W_1$ is submersive. Because $X_k \setminus (\tilde{\phi}^{-1}(W_1) \cup \rho^{-1}(G)) \rightarrow \overline{Y}_k^{\text{sm}} \setminus (\phi^{-1}(W_1) \cup G)$ is a local diffeomorphism, the morphism $\overline{Y}_k^{\text{sm}} \setminus (\phi^{-1}(W_1) \cup G) \rightarrow \phi(Y) \setminus W_1$ is submersive.

If $\dim \phi(\overline{Y}_k^{\text{sm}}) < \ell$, let W_1 denote the closure of $\phi(\overline{Y}_k^{\text{sm}})$ in $\phi(Y)$, and we set $G := \emptyset$. Then, we clearly have that $\overline{Y}_k^{\text{sm}} \setminus (\phi^{-1}(W_1) \cup G) = \emptyset \rightarrow \phi(Y) \setminus W_1$ is submersive.

We have $\dim(G \cup Y') \leq k - 1$. Suppose that $\dim \phi(G \cup Y') = \ell$. By using the hypothesis of the induction, we have a closed subanalytic subset $W_2 \subset \phi(G \cup Y')$ such that (i) $\dim W_2 < \ell$, (ii) we have $\dim((G \cup Y') \cap \phi^{-1}(P)) \leq k - \ell - 1$ for any $P \in \phi(G \cup Y') \setminus W_2$. If $\dim \phi(G \cup Y') < \ell$, we set $W_2 := \phi(G \cup Y')$. Then, we clearly have $\dim((G \cup Y') \cap \phi^{-1}(P)) \leq k - \ell - 1$ for any $P \in \phi(G \cup Y') \setminus W_2 = \emptyset$.

Let W denote the closure of the union of W_1 and W_2 . Then, the claim of the lemma holds. ■

2.2.2 Ramified analyticity at boundary points

Let U be a subanalytic open subset in an n -dimensional real analytic manifold M . Let f be an analytic function on U which is subanalytic on (U, M) . Let ∂U denote the boundary of U , which is $(n - 1)$ -dimensional. Let P be any point of ∂U at which ∂U is smooth. We say that f is ramified analytic around P if the following holds:

- Let $(\mathcal{U}, t_1, \dots, t_n)$ be a real analytic coordinate neighbourhood of \mathbb{R}^n around P such that $\mathcal{U} \cap \partial U = \{t_1 = 0\}$. Then, f is expressed as

$$f = \sum_{j \geq -N_1} a_j(t_2, \dots, t_n) t_1^{j/\rho}$$

where ρ is a positive integer, and a_j are analytic functions.

For any subanalytic set A , let $\text{Sing}(A)$ denote the singular locus of A .

Lemma 2.12 *Let f be any continuous function on U which is subanalytic on (U, M) . We have a closed $(n - 2)$ -dimensional subset $Z \subset \partial U$ with the following property.*

- $Z \supset \text{Sing}(\partial U)$.
- f is ramified analytic around any point $P \in \partial U \setminus Z$.

Proof We have a rectilinearization (W_α, ϕ_α) ($\alpha \in \Lambda$) of (U, f) and ∂U . We have compact subsets $K_\alpha \subset W_\alpha$ such that $\bigcup \phi_\alpha(K_\alpha) = \mathbb{R}^n$. Let T_α denote the union of quadrants Q in W_α such that (i) $Q \subset \phi_\alpha^{-1}(\partial U)$, (ii) $\dim Q \leq n-2$. Each ϕ_α is factorized as in (1) of §2.1.1. Let $\psi_\alpha^{(\ell)} : W_\alpha^{(\ell)} \rightarrow \mathbb{R}^n$ denote the induced map. We have the closed subanalytic subsets $Z_\alpha^{(\ell)} := \phi_\alpha(K_\alpha) \cap \psi_\alpha^{(\ell)}(C_\alpha^{(\ell)})$. We set

$$Z = \left(\bigcup_{\alpha} \phi_\alpha(K_\alpha \cap T_\alpha) \right) \cup \left(\partial U \cap \bigcup_{\alpha, \ell} Z_\alpha^{(\ell)} \right) \cup \text{Sing}(\partial U).$$

Let $P \in \partial U \setminus Z$. We have $\alpha \in \Lambda$ such that $P \in \phi_\alpha(K_\alpha)$. By our choice of Z , we have a neighbourhood X_P of P in \mathbb{R}^n such that ϕ_α induces a diffeomorphism $\phi_\alpha^{-1}(X_P) \simeq X_P$. Let (s_1, \dots, s_n) be the coordinate of W_α . Because ∂U is smooth around P , we may assume that $\phi_\alpha^{-1}(X_P \cap U) = \{s_1 > 0\} \cap \phi_\alpha^{-1}(X_P)$. Note that $\phi_\alpha^*(f)$ is analytic on $\{s_1 > 0\} \cap \phi_\alpha^{-1}(X_P)$, and that $\phi_\alpha^*(f) \in \text{RNC}(Q)$ for any $Q \in \text{Quad}_n(\phi_\alpha^{-1}(U), W_\alpha)$. Hence, we have the expression $\phi_\alpha^*(f) = \sum a_j(s_2, \dots, s_n)s_1^{j/\rho}$ on $\phi_\alpha^{-1}(X_P \cap U)$. ■

Let us consider the case $M = M_0 \times \mathbb{R}$, and U is contained in $M_0 \times \{x_n > 0\}$. Here, M_0 is an $(n-1)$ -dimensional real analytic manifold. We set $V := \overline{U} \cap (M_0 \times \{0\})$. Let V° be the set of interior points of V as a subset of $M_0 \times \{0\}$. Let \overline{V}° denote the closure of V° in $M_0 \times \{0\}$. Let $\partial(V^\circ)$ denote the boundary of V° as a subset of $M_0 \times \{0\}$. Let f be a continuous subanalytic function on (U, M) . By Lemma 2.12, we have the following.

Lemma 2.13 *We have a closed subanalytic subset $Z_0 \subset \overline{V}^\circ$ such that the following holds:*

- We have $\dim Z_0 \leq n-2$ and $Z_0 \supset \partial V^\circ$.
- For any $P \in V^\circ \setminus Z_0$, we have a neighbourhood X_P in M such that $U \cap X_P = X_P \cap \{x_n > 0\} =: U_P$ and that $f|_{U_P}$ is expressed as $\sum \alpha_j(x_1, \dots, x_{n-1})x_n^{j/\rho}$, where ρ is a positive integer, and α_j are analytic functions. ■

Moreover, we have the following.

Lemma 2.14 *We have a closed subanalytic subset $Z_1 \subset Z_0$ with the following property:*

- We have $\dim Z_1 \leq n-3$, $Z_1 \supset \text{Sing}(\partial V^\circ)$, and $Z_1 \supset \text{Sing}(Z_0)$.
- Let P be any point of $\partial V^\circ \setminus Z_1$. We take a real analytic local coordinate neighbourhood $(\mathcal{U}_P, y_1, \dots, y_{n-1})$ in $M_0 \times \{0\}$ around P such that $\mathcal{U}_P \cap V^\circ = \{y_{n-1} > 0\}$. Then, we have a positive integer $m > 0$ and a positive number $C > 0$ such that

$$\mathcal{B} := \{(y_1, \dots, y_{n-1}, x_n) \mid 0 < x_n < Cy_{n-1}^m\} \subset U.$$

Moreover, $f|_{\mathcal{B}}$ is analytic with respect to $(y_1, \dots, y_{n-2}, y_{n-1}^{1/\rho}, (y_{n-1}^{-m}x_n)^{1/\rho})$ for a positive integer ρ , i.e., $f|_{\mathcal{B}}$ is expressed as a convergent power series

$$f|_{\mathcal{B}} = \sum_{i \geq -N_1} \sum_{j \geq -N_2} A_{ij}(y_1, \dots, y_{n-2}) \cdot y_{n-1}^{i/\rho} \cdot (y_{n-1}^{-m}x_n)^{j/\rho}.$$

- Let P be any point of $(V^\circ \cap Z_0) \setminus Z_1$. We take a real analytic coordinate neighbourhood $(\mathcal{U}_P, y_1, \dots, y_{n-1})$ of $M_0 \times \{0\}$ such that $\mathcal{U}_P \cap (V^\circ \setminus Z_0) = \{y_{n-1} \neq 0\}$ around P . Then, we have a positive integer $m > 0$ and a positive number $C > 0$ such that

$$\mathcal{B}_+ := \{(y_1, \dots, y_{n-1}, x_n) \mid y_{n-1} > 0, 0 < x_n < Cy_{n-1}^m\} \subset U,$$

$$\mathcal{B}_- := \{(y_1, \dots, y_{n-1}, x_n) \mid y_{n-1} < 0, 0 < x_n < C(-y_{n-1})^m\} \subset U.$$

Moreover, f is analytic with respect to $(y_1, \dots, y_{n-2}, (\pm y_{n-1})^{1/\rho}, ((\pm y_{n-1})^{-m}x_n)^{1/\rho})$ on \mathcal{B}_\pm for a positive integer ρ , i.e., $f|_{\mathcal{B}_\pm}$ is expressed as a convergent power series

$$f|_{\mathcal{B}_\pm} = \sum_{i \geq -N_1} \sum_{j \geq -N_2} A_{\pm, ij}(y_1, \dots, y_{n-2}) \cdot (\pm y_{n-1})^{i/\rho} \cdot ((\pm y_{n-1})^{-m}x_n)^{j/\rho}.$$

If U is relatively compact in M , we have the boundedness of the positive numbers ρ and m in the second and third properties.

Proof It is enough to study the connected components \mathcal{C} of $U \cap \left((M_0 \times \{0\} \setminus Z_0) \times \mathbb{R}_{\geq 0} \right)$ and the restriction of f to \mathcal{C} . Note that the third property for (U, f) follows from the second properties of $(\mathcal{C}, f|_{\mathcal{C}})$ for any connected components \mathcal{C} . Hence, we may assume $Z_0 = \partial V^\circ$ from the beginning.

We have a rectilinearization (W_α, ϕ_α) ($\alpha \in \Lambda$) of (U, f) . We may also assume that $\{(W_\alpha, \phi_\alpha)\}$ is a rectilinearization of V° and ∂V° , and that $\phi_\alpha^*(x_n)$ are normal crossing for any $\alpha \in \Lambda$. We take compact subsets $K_\alpha \subset W_\alpha$ such that $M = \bigcup \phi_\alpha(K_\alpha)$.

For each α , let T_α denote the union of quadrants Q in W_α such that (i) $Q \subset \phi_\alpha^{-1}(\partial V^\circ)$, (ii) $\dim \phi_\alpha(Q) \leq n-3$.

For a fixed α , we have a factorization of ϕ_α into a finite sequence of local real blowings up as in (1) in §2.1.1. Let $\psi_\alpha^{(\ell)} : W_\alpha^{(\ell)} \rightarrow \mathbb{R}^n$ be the induced map. Let us look at the strict transforms $(\overline{V^\circ})_\alpha^{(\ell)} \subset W_\alpha^{(\ell)}$ of $\overline{V^\circ}$. Note that $\dim C_\alpha^{(\ell)} \leq n-2$. If $\dim(C_\alpha^{(\ell)} \cap \partial(\overline{V^\circ})_\alpha^{(\ell)}) < n-2$, let $Z_\alpha^{(\ell)}$ denote the closure of $\psi_\alpha^{(\ell)}(C_\alpha^{(\ell)} \cap \partial(\overline{V^\circ})_\alpha^{(\ell)}) \cap \phi_\alpha(K_\alpha)$. If $\dim(C_\alpha^{(\ell)} \cap \partial(\overline{V^\circ})_\alpha^{(\ell)}) = n-2$, let $Z_\alpha^{(\ell)}$ denote the closure of $\psi_\alpha^{(\ell)}(\text{Sing}(C_\alpha^{(\ell)} \cap \partial(\overline{V^\circ})_\alpha^{(\ell)})) \cap \phi_\alpha(K_\alpha)$. Let Z_1 denote the union of $\text{Sing}(\partial \overline{V^\circ}) = \text{Sing}(Z_0)$, $\phi_\alpha(K_\alpha \cap T_\alpha)$ for all α , and $Z_\alpha^{(\ell)}$ for all of α and ℓ . By construction, Z_1 is a subanalytic subset with $\dim Z_1 \leq n-3$.

Let us study the second property. Let P be any point of $\partial V^\circ \setminus Z_1$. We take a coordinate neighbourhood $(\mathcal{U}_P, y_1, \dots, y_{n-1})$ of P in M_0 such that $V^\circ \cap \mathcal{U}_P = \{y_{n-1} > 0\}$. By shrinking \mathcal{U}_P , we may assume that $\mathcal{U}_P = \{(y_1, \dots, y_{n-1}) \mid |y_i| < \epsilon\}$ for $\epsilon > 0$. Let X_P be a neighbourhood P in $M = M_0 \times \mathbb{R}$ given by $\mathcal{U}_P \times \{|x_n| < \epsilon\}$, which is the product of $X_{1,P} := \{(y_1, \dots, y_{n-2}) \mid |y_i| < \epsilon\}$ and $X_{2,P} := \{(y_{n-1}, x_n) \mid |y_{n-1}| < \epsilon, |x_n| < \epsilon\}$.

We construct a sequence of blowings up inductively. Set $Y^{(0)} := \mathbb{R}^2 = \{(y_{n-1}, x_n)\}$, $H^{(0)} := \{x_n = 0\}$, and $Q^{(0)} := (0, 0)$. Let $\kappa_1 : Y^{(1)} \rightarrow Y^{(0)}$ be the real blowing up at $Q^{(0)}$. Let $H^{(1)}$ denote the strict transform of $H^{(0)}$, and set $Q^{(1)} := \kappa_1^{-1}(Q^{(0)}) \cap H^{(1)}$. Suppose that we are given a sequence of morphisms

$$Y^{(i)} \xrightarrow{\kappa_i} Y^{(i-1)} \xrightarrow{\kappa_{i-1}} \dots \xrightarrow{\kappa_1} Y^{(0)}$$

with points $Q^{(j)} \in Y^{(j)}$ such that κ_{j+1} are the real blowing up at $Q^{(j)}$, and $H^{(j)} \subset Y^{(j)}$ which is the strict transform of $H^{(0)}$. Then, let $\kappa_{i+1} : Y^{(i+1)} \rightarrow Y^{(i)}$ be the blowing up at $Q^{(i)}$. Let $H^{(i+1)}$ be the strict transform of $H^{(i)}$ with respect to κ_{i+1} . We set $Q^{(i+1)} := \kappa_{i+1}^{-1}(Q^{(i)}) \cap H^{(i+1)}$. Thus, the inductive construction can proceed. For each i , we have the natural coordinate neighbourhood around $Q^{(i)}$ given by $(y_{n-1}, x_n y_{n-1}^{-i})$. We have the induced map $\nu_i : X_{1,P} \times Y^{(i)} \rightarrow X_{1,P} \times Y^{(0)}$.

We take a path $\gamma : [0, \delta] \rightarrow \overline{V^\circ}$ such that $\gamma([0, \delta] \setminus \{0\}) \subset V^\circ$ and $\gamma(0) = P$. We may assume that $\gamma([0, \delta] \setminus \{0\})$ does not intersect with the set of the critical values of ϕ_α for any $\alpha \in \Lambda$. After making δ smaller, we have $\alpha_0 \in \Lambda$ such that $\gamma([0, \delta]) \subset \phi_{\alpha_0}(K_{\alpha_0})$. We have the path $\tilde{\gamma}_\alpha$ to W_α such that $\phi_\alpha \circ \tilde{\gamma}_\alpha = \gamma$. By the construction of Z_1 , we have a number k and an open subset $\mathcal{U}_k \subset \nu_k^{-1}(X_P)$ such that $\phi_\alpha^{-1}(X_P) \rightarrow X_P$ is identified with $\nu_k|_{\mathcal{U}_k} : \mathcal{U}_k \rightarrow X_P$ around the image of $\tilde{\gamma}_\alpha$. After making ϵ smaller, we may assume that $X_{1,P} \times \{(y_{n-1}, x_n) \in X_{2,P} \mid y_{n-1} > 0, x_n = 0\}$ is contained in $\phi_\alpha(W_\alpha)$.

Let (s_1, \dots, s_n) be the coordinate of W_α . Because $\phi_\alpha^{-1}(V^\circ)$ is the union of the quadrants contained in $\phi_\alpha^{-1}(V^\circ)$, we may assume that $\phi_\alpha^{-1}(V^\circ \cap X_P) = \{s_n = 0, s_{n-1} > 0\} \cap \phi_\alpha^{-1}(X_P)$. We have the analytic functions $b_1 := \phi_\alpha^*(x_n y_{n-1}^{-k})$ and $b_2 := \phi_\alpha^*(y_{n-1})$ on $\phi_\alpha^{-1}(X_P)$. We have the analytic function s_n on $\phi_\alpha^{-1}(X_P)$. Both of the differentials db_1 and ds_n are nowhere vanishing on $\phi_\alpha^{-1}(X_P)$. Note that $\phi_\alpha^{-1}(V^\circ \cap X_P) \subset b_1^{-1}(0) \cap s_n^{-1}(0)$. Then, we obtain that $A_0 := s_n/b_1$ is an analytic function on $\phi_\alpha^{-1}(X_P)$, which is nowhere vanishing. We may assume that $A_0 > 0$. We have $\phi_\alpha^{-1}(X_P \cap V^\circ) = \{s_n = 0\} \cap \{b_2 > 0\} \cap \phi_\alpha^{-1}(X_P)$. The derivatives db_2 and ds_{n-1} are nowhere vanishing on $\{s_n = 0\} \cap \phi_\alpha^{-1}(X_P)$. Hence, on $\{s_n = 0\} \cap \phi_\alpha^{-1}(X_P)$, s_{n-1}/b_2 is an analytic function and positive. Hence, after shrinking X_P , we have $s_{n-1} = A_1 b_2 + B \cdot b_1$ on $\phi_\alpha^{-1}(X_P)$, where A_1 is a nowhere vanishing analytic function on $\phi_\alpha^{-1}(X_P)$, and B is an analytic function on $\phi_\alpha^{-1}(X_P)$.

Because $\phi_\alpha^{-1}(V^\circ \cap X_P)$ contains $\phi_\alpha^{-1}(X_P) \cap \{s_n = 0, s_{n-1} > 0\}$, and because $\phi_\alpha^{-1}(V^\circ)$ and $\phi_\alpha^{-1}(U)$ are rectilinearized, we obtain that $\phi_\alpha^{-1}(U \cap X_P)$ contains $\phi_\alpha^{-1}(X_P) \cap \{s_n > 0, s_{n-1} > 0\}$. We may assume that $T_\alpha \cap \phi_\alpha^{-1}(X_P) = \emptyset$, where T_α is as in the proof of Lemma 2.12. Hence, s_i ($i < n-1$) are nowhere vanishing on $\phi_\alpha^{-1}(X_P)$. Then, f is expressed as $\sum c_{j_1, j_2}(s_1, \dots, s_{n-2}) s_n^{j_1/\rho} s_{n-1}^{j_2/\rho}$ on $\phi_\alpha^{-1}(X_P) \cap \{s_n > 0, s_{n-1} > 0\}$. Then, the second property is clear. \blacksquare

2.2.3 Fibration and ramified analyticity

Let M_1 be an $(n-2)$ -dimensional real analytic manifold. Set $A := \mathbb{R}_{\geq 0} \times M_1$ and $B := S^1 \times A$. We have $\partial B = S^1 \times \partial A$. Let U be a subanalytic open subset in $B \setminus \partial B$ with an analytic function f on U , which is subanalytic on (U, B) . We have $\dim U = n$. We set $V := \overline{U} \cap \partial B$. Let V° denote the set of the interior points of $V \subset \partial B$. We have $\dim V^\circ = n-1$ if it is not empty. We have a closed subanalytic subset $Z_1 \subset Z_0 \subset \overline{V^\circ}$ as in Lemma 2.13 and Lemma 2.14.

Let $q : B \rightarrow A$ be the projection. We set $W := q(\overline{V^\circ}) \subset \partial A$. We have $\dim W \leq \dim \partial A = n-2$. We have $q(Z_1) \subset q(Z_0) \subset W$. We have $\dim q(Z_1) \leq n-3$ and $\dim q(Z_0) \leq n-2$.

Lemma 2.15 *We have a closed subanalytic subset $W_0 \subset W$ with the following property.*

- We have $\dim W_0 \leq n-3$.
- $\partial W \subset W_0$.
- $Z_0 \setminus q^{-1}(W_0)$ is horizontal with respect to q , i.e., $\dim q^{-1}(P) = 0$ for any $P \in W \setminus W_0$. ■

Because $\dim q^{-1}(W_0) \leq n-2$, we may assume that Z_0 contains $Y_0 := q^{-1}(W_0) \cap \overline{V^\circ}$, by enlarging Z_0 and Z_1 . Let Y_1 denote the closure of $Z_0 \setminus Y_0$ in Z_0 . We have $\dim(Y_0 \cap Y_1) \leq n-3$. We may assume that Z_1 contains $Y_0 \cap Y_1$ by enlarging Z_1 .

Lemma 2.16 *We have a closed subanalytic subset $W_1 \subset W_0$ and a closed subset $G \subset \overline{(Y_0)_{n-2}^{\text{sm}}}$ with the following property.*

- We have $\dim W_1 \leq n-4$ and $\dim G \leq n-3$.
- $(Z_1 \cup G) \setminus q^{-1}(W_1)$ is horizontal.
- $(Y_0)_{n-2}^{\text{sm}} \setminus (G \cup q^{-1}(W_1)) \rightarrow W_0 \setminus W_1$ is submersive.

Proof It follows from Lemma 2.11. ■

Let r be the coordinate of $\mathbb{R}_{\geq 0}$, and θ be the local coordinate of $S^1 = \mathbb{R}/2\pi\mathbb{Z}$. Let (x_1, \dots, x_{n-2}) be the coordinate of \mathbb{R}^{n-2} . We set $R_0 := Z_0$ and $R_1 := Z_1 \cup G$. The following lemma is clear by construction.

Lemma 2.17 *We have $\dim R_0 \leq n-2$, $\dim R_1 \leq n-3$, $\dim W_0 \leq n-3$ and $\dim W_1 \leq n-4$. We have $R_1 \subset q^{-1}(W_0)$. The following holds:*

- $R_0 \setminus q^{-1}(W_0)$ and $R_1 \setminus q^{-1}(W_1)$ are horizontal with respect to q .
- Take $P \in W \setminus W_0$ such that $\dim(q^{-1}(P) \cap \overline{V^\circ}) = 1$. Let Q be any interior point of $q^{-1}(P) \cap \overline{V^\circ} \setminus R_0 \subset q^{-1}(P)$. Then, we have a neighbourhood \mathcal{U} of Q in \overline{U} and a positive integer ρ such that the restriction of f to \mathcal{U} is expressed as

$$f = \sum_{j \geq -N_1} \alpha_j(\theta, x_1, \dots, x_{n-2}) \cdot r^{j/\rho}.$$

Here, α_j are analytic functions.

- Take $P \in W_0 \setminus (W_1 \cup \partial W)$ such that $\dim(q^{-1}(P) \cap \overline{V^\circ}) = 1$. Let Q be any interior point of $q^{-1}(P) \cap \overline{V^\circ} \setminus R_1 \subset q^{-1}(P)$. We have a coordinate neighbourhood $(\mathcal{N}; y_1, \dots, y_{n-2})$ around P in \mathbb{R}^{n-2} , such that $W_0 \cap \mathcal{U} = \{y_{n-2} = 0\}$. We take real numbers $\theta_1 < \theta_2$ the interval $[\theta_1, \theta_2]$ is a small neighbourhood of Q in $q^{-1}(P) \cap \overline{V^\circ}$ such that $[\theta_1, \theta_1] \cap R_1 = \emptyset$. Then, we have a positive integer ρ , a positive integer m , and a positive number $C > 0$, such that

$$\mathcal{U}_\pm = \{(y_1, \dots, y_{n-2}, \theta, r) \mid (y_1, \dots, y_{n-2}) \in \mathcal{N}, \theta_1 < \theta < \theta_2, 0 < r < C(\pm y_{n-2})^m\} \subset U$$

and that the restriction of f to \mathcal{U}_\pm are expressed as

$$f = \sum_{i \geq -N_1} \sum_{j \geq -N_2} \alpha_{\pm, i, j}(y_1, \dots, y_{n-3}, \theta) \cdot y_{n-2}^i \cdot (y_{n-2}^{-m} r)^j$$

Here $\alpha_{\pm, i, j}$ are analytic functions.

- Take $P \in \partial W \setminus W_1$ such that $\dim(q^{-1}(P) \cap \overline{V^\circ}) = 1$. Let Q be any interior point of $q^{-1}(P) \cap \overline{V^\circ} \setminus R_1 \subset q^{-1}(P)$. We have a coordinate neighbourhood $(\mathcal{N}; y_1, \dots, y_{n-2})$ around P in \mathbb{R}^{n-2} such that $W \cap \mathcal{N} = \{y_{n-2} \geq 0\}$. We take real numbers $\theta_1 < \theta_2$ such that $[\theta_1, \theta_2]$ is a neighbourhood of Q in $q^{-1}(P) \cap \overline{V^\circ}$ such that $[\theta_1, \theta_1] \cap R_1 = \emptyset$. Then, we have a positive integer ρ , a positive integer m , and a positive number $C > 0$, such that

$$\mathcal{U} = \{(y_1, \dots, y_{n-2}, \theta, r) \mid (y_1, \dots, y_{n-2}) \in \mathcal{N}, 0 < y_{n-2}, \theta_1 < \theta < \theta_2, 0 < r < Cy_{n-2}^m\} \subset U$$

and that the restriction of f to \mathcal{U} is expressed as

$$f = \sum_{i \geq -N_1} \sum_{j \geq -N_2} \alpha_{i,j}(y_1, \dots, y_{n-3}, \theta) \cdot y_{n-2}^i \cdot (y_{n-2}^{-m} r)^j.$$

Here, $\alpha_{i,j}$ are analytic functions. ■

2.2.4 Lift of maps at boundary

Let $F : N \rightarrow M$ be a real analytic map of real analytic manifolds. Let U be a subanalytic relatively compact open subset of N . Let W be a real analytic manifold equipped with an isomorphism $W \simeq \mathbb{R}^m$. Let $\phi : W \rightarrow M$ be a real analytic map. Suppose that we are given a continuous map $g : U \rightarrow W$ such that (i) $F|_U = \phi \circ g$, (ii) we have a compact subset $K \subset W$ such that the graph of g is contained in $N \times K$, (iii) g is subanalytic in the sense that the graph of g is subanalytic in $N \times W$.

Lemma 2.18 *We have a closed subanalytic subset $Z \subset \partial U$ with $\dim Z \leq \dim N - 2$ such that (i) Z contains the singular locus of ∂U , (ii) we have a continuous subanalytic map $\bar{g} : \overline{U} \setminus Z \rightarrow W$ such that $\bar{g}|_U = g$.*

Proof We are given the coordinate (y_1, \dots, y_n) of W . We have the subanalytic functions $g^*(y_i)$ on U . They are bounded. By Lemma 2.12, we have a closed subanalytic subset $Z \subset \partial U$ with $\dim Z \leq \dim N - 2$ such that $g^*(y_i)$ are ramified analytic around any point of $\partial U \setminus Z$. Then, the claim is clear. ■

Lemma 2.19 *Suppose that ϕ is obtained as the composition of a finite sequence of local real blowings up. Then, the map \bar{g} in Lemma 2.18 is analytic. Here, we regard $\overline{U} \setminus Z$ as a real analytic manifold with smooth boundary.*

Proof We take any point $P \in \partial U \setminus Z$. We take a coordinate $(N_P; y_1, \dots, y_n)$ of N around P such that $N_P \cap \partial U = \{y_n = 0\}$ and $N_P \cap U = \{y_n > 0\}$. We have a positive integer ρ and the description $\bar{g}^*(x_j) = \sum_{k \geq 0} a_{j,k}(y_1, \dots, y_{n-1}) y_n^{k/\rho}$. Fix $Q = (y_1, y_2, \dots, y_{n-1}, 0) \in N_P$. We consider the path $\gamma_Q : [0, 1] \rightarrow N_P$ given by $\gamma_Q(t) = (y_1, \dots, y_{n-1}, t)$. Because ϕ is assumed to be the composition of a finite sequence of local real blowings up, and because the image of γ_Q is contained in the image of W , we can observe that we have a real analytic map $\tilde{\gamma}_Q : [0, 1] \rightarrow W$ such that $\phi \circ \tilde{\gamma}_Q = \gamma_Q$. It implies that $\gamma_Q^* \bar{g}^*(x_j)$ are real analytic functions on $[0, 1]$. Hence, we obtain that $\rho = 1$, i.e., \bar{g} is analytic around P . ■

2.2.5 Boundedness

Let M be a real analytic manifold. Let $I := \{0 \leq t \leq 1\}$. Let $\pi : M \times I \rightarrow M$ be the projection. Let B be any subanalytic subset in $M \times [0, 1]$ such that $\dim B = \dim M + 1$. For any $x \in M$, let $B_x := B \cap (\{x\} \times [0, 1])$. Let $\overline{B_x}$ denote the closure of B_x .

Lemma 2.20 *Suppose that $\overline{B_x}$ does not contain $(x, 1)$ for any $x \in M$. Then, we have a subanalytic closed subset $Z \subset M$ such that the following holds.*

- $\dim Z < \dim M$.
- The closure of $B \cap ((M \setminus Z) \times [0, 1])$ in $(M \setminus Z) \times [0, 1]$ does not intersect with $(M \setminus Z) \times \{1\}$.

Proof Let \overline{B} denote the closure of B in $M \times [0, 1]$. Let B° denote the interior part of B . We have the closed subanalytic subset $R := \overline{B} \setminus B^\circ$. We have $\dim R \leq \dim M$. Hence, we have a closed subanalytic subset $Z_1 \subset M$ such that (i) $\dim Z_1 < \dim M$, (ii) each connected component of $M \setminus Z_1$ is simply connected, (iii) the induced map $R \setminus \pi^{-1}(Z_1) \rightarrow M \setminus Z_1$ is proper and a local diffeomorphism. On each connected component \mathcal{C} of $M \setminus Z_1$, we have subanalytic functions $h_i^\mathcal{C}$ on (\mathcal{C}, M) such that $\pi^{-1}(\mathcal{C}) \cap R$ is the union of the graph of $h_i^\mathcal{C}$. We may assume $h_i^\mathcal{C} < h_{i+1}^\mathcal{C}$. Then, $\overline{B} \cap \pi^{-1}(\mathcal{C})$ is the union of the set of the form $\{(x, t) \mid x \in \mathcal{C}, h_i(x) \leq t \leq h_{i+1}(x)\}$. We also have $\pi^{-1}(\mathcal{C}) \cap (\overline{B} \setminus B)$ is relatively 0-dimensional over \mathcal{C} . Then, the claim of the lemma is clear. \blacksquare

Let Y_i ($i = 1, 2$) be real analytic manifolds. Let \mathcal{A} be a relatively compact subanalytic subset of $Y_1 \times Y_2$. Let f be a subanalytic function on $(\mathcal{A}, Y_1 \times Y_2)$. Let $q : Y_1 \times Y_2 \rightarrow Y_2$ denote the projection.

Lemma 2.21 *Suppose that $f|_{\mathcal{A} \cap q^{-1}(x)}$ is bounded for any $x \in Y_2$. Then, we have a closed subanalytic subset $Z \subset Y_2$ such that (i) $\dim Z < \dim Y_2$, (ii) for any $x \in Y \setminus Z$ we have a neighbourhood U_x of x in $Y \setminus Z$ such that $f|_{\mathcal{A} \cap q^{-1}(U_x)}$ is bounded.*

Proof We set $\overline{\mathbb{R}} := \mathbb{R} \cup \{\pm\infty\}$. Let $\Gamma_f \subset Y_1 \times Y_2 \times \overline{\mathbb{R}}$ denote the graph of f . Let B_f denote the image of Γ_f by the projection $Y_1 \times Y_2 \times \overline{\mathbb{R}} \rightarrow Y_2 \times \overline{\mathbb{R}}$. It is a subanalytic subset. Then, we obtain the claim of this lemma from Lemma 2.20. \blacksquare

2.3 Meromorphic flat bundles

2.3.1 One dimensional case

Let $X := \Delta$ and $H := \{0\}$. For any positive integer e , set $X^{(e)} := \Delta_{\zeta_e}$ and $H^{(e)} := \{0\}$. Let $\varphi_e : (X^{(e)}, H^{(e)}) \rightarrow (X, H)$ be the ramified covering given by $\varphi_e(\zeta_e) = \zeta_e^e$. Let (V, ∇) be a meromorphic flat bundle on (X, H) . According to the Hukuhara-Levelt-Turrittin theorem, we have a positive integer e and a set $\text{Irr}(V, \nabla) \subset \mathcal{O}_{X^{(e)}}(*H^{(e)})/\mathcal{O}_{X^{(e)}}$ and a decomposition

$$\varphi_e^*(V, \nabla)|_{\widehat{H^{(e)}}} = \bigoplus_{\mathfrak{a} \in \text{Irr}(V, \nabla)} (\widehat{V}_{\mathfrak{a}}, \nabla_{\mathfrak{a}})$$

such that $\nabla_{\mathfrak{a}} - d\mathfrak{a} \text{id}_{\widehat{V}_{\mathfrak{a}}}$ are regular singular. The set $\text{Irr}(V, \nabla)$ is invariant under the action of the Galois group of the ramified covering $X^{(e)} \rightarrow X$. In this paper, it is called the set of irregular values of (V, ∇) . The ranks $\text{rank } \widehat{V}_{\mathfrak{a}}$ are called the multiplicity of \mathfrak{a} . They give a map $\text{rank} : \text{Irr}(V, \nabla) \rightarrow \mathbb{Z}_{>0}$, called the multiplicity function. Set $\widehat{R} := \varprojlim \mathbb{C}[[z^{1/e}]]$ and $\widehat{K} := \varprojlim \mathbb{C}((z^{1/e}))$. We may naturally regard $\text{Irr}(V, \nabla)$ as a subset in \widehat{K}/\widehat{R} . If we have $\text{Irr}(V, \nabla) \subset \mathbb{C}((z))/\mathbb{C}[[z]]$, then (V, ∇) is called unramified. We shall often use the natural bijection $\zeta_e^{-1}\mathbb{C}[\zeta_e^{-1}] \simeq \mathbb{C}((\zeta_e))/\mathbb{C}[[\zeta_e]]$.

Let $\varpi : \widetilde{X}(H) \rightarrow X$ be the oriented real blowing up. A C^∞ -function f on an open subset $\mathcal{U} \subset \widetilde{X}(H)$ is called holomorphic if $f|_{\mathcal{U} \cap \varpi^{-1}(H)}$ is holomorphic. Let $\mathcal{O}_{\widetilde{X}(H)}$ denote the sheaf of holomorphic functions on $\widetilde{X}(H)$. For any \mathcal{O}_X -module \mathcal{M} , let $\varpi^*\mathcal{M} := \varpi^{-1}\mathcal{M} \otimes_{\varpi^{-1}\mathcal{O}_X} \mathcal{O}_{\widetilde{X}(H)}$. According to a classical theory, for any $Q \in \varpi^{-1}(H)$, we have a neighbourhood \mathcal{U}_Q of Q in $\widetilde{X}(H)$ and a decomposition

$$\varpi^*(V, \nabla)|_{\mathcal{U}_Q} \simeq (V_{\mathfrak{a}, \mathcal{U}_Q}, \nabla_{\mathfrak{a}, \mathcal{U}_Q}) \quad (2)$$

such that $(V_{\mathfrak{a}, \mathcal{U}_Q}, \nabla_{\mathfrak{a}, \mathcal{U}_Q})|_{\varpi^{-1}(\widehat{H}) \cap \mathcal{U}_Q} = \bigoplus_{\mathfrak{a}} \varpi^*(\widehat{V}_{\mathfrak{a}}, \nabla_{\mathfrak{a}})$. Such a decomposition is not uniquely determined. We set $\mathcal{F}_{\mathfrak{a}}^{\mathcal{U}_Q}(\varpi^*V) = \bigoplus_{\mathfrak{b} \leq \mathfrak{a}} V_{\mathfrak{b}, \mathcal{U}_Q}$. Then, the filtration \mathcal{U}_Q is well defined. Such a filtration $\mathcal{F}^{\mathcal{U}_Q}$ is called Stokes filtration.

Let L be the local system on $X \setminus H$, which is the sheaf of flat sections of (V, ∇) . Let \widetilde{L} be the local system on $\widetilde{X}(H)$ induced by L . Let \widetilde{L}_Q denote the stalk of \widetilde{L} at Q . Suppose that (V, ∇) is unramified. We choose a frame \mathbf{v} of V , and let h_V be a metric determined by $h_V(v_i, v_j) = \delta_{i,j}$. Then, we have the filtration \mathcal{F}^Q of \widetilde{L}_Q indexed by the partially ordered set $(z^{-1}\mathbb{C}[[z^{-1}]], \leq_Q)$ determined by the following condition.

- $s \in \mathcal{F}_c^Q(\tilde{L}_Q)$ if $|s \cdot \exp(\mathbf{c})|_h = O(|z|^{-C})$.

Set $\text{Gr}_c^{\mathcal{F}^Q}(\tilde{L}_Q) := \mathcal{F}_c^Q(\tilde{L}_Q) / \mathcal{F}_{<c}^Q(\tilde{L}_Q)$. By the existence of a decomposition (2), we have $\text{Gr}_c^{\mathcal{F}^Q}(\tilde{L}_Q) \neq 0$ if and only if $\mathbf{c} \in \text{Irr}(V, \nabla)$. We also have a decomposition $\tilde{L}_Q = \bigoplus G_{Q,\mathbf{a}}$ such that $\mathcal{F}_a^Q(\tilde{L}_Q) = \bigoplus_{\mathbf{b} \leq_Q \mathbf{a}} G_{\mathbf{a}}$.

If Q' is sufficiently close, then the identity on $z^{-1}\mathbb{C}[z^{-1}]$ induces a morphism of partially ordered set $(\text{Irr}(V, \nabla), \leq_Q) \rightarrow (\text{Irr}(V, \nabla), \leq_{Q'})$. We have the isomorphism $\tilde{L}_Q \simeq \tilde{L}_{Q'}$. Any splitting of \mathcal{F}^Q gives a splitting of $\mathcal{F}^{Q'}$. In this sense, the family $\{\mathcal{F}^Q \mid Q \in \varpi^{-1}(H)\}$ satisfies a compatibility condition. It is called the Stokes structure of (V, ∇) .

Let $\mathbf{a}, \mathbf{b} \in \text{Irr}(V, \nabla)$. We obtain a C^∞ -function $F_{\mathbf{a},\mathbf{b}} := |z|^{-\text{ord}(\mathbf{a}-\mathbf{b})}(\mathbf{a} - \mathbf{b})$ on $\tilde{X}(H)$. Let $Z(\mathbf{a}, \mathbf{b}) := \{Q \in \varpi^{-1}(H), \mid F_{\mathbf{a},\mathbf{b}}(Q) = 0\}$.

The following lemma is well known.

Lemma 2.22 *Let I be an open interval in $\varpi^{-1}(H)$. Suppose that $|Z(\mathbf{a}, \mathbf{b}) \cap I| \leq 1$ for any pair $(\mathbf{a}, \mathbf{b}) \in \text{Irr}(V, \nabla)$ with $\mathbf{a} \neq \mathbf{b}$. Then, we have a decomposition $L|_I = \bigoplus_{\mathbf{a} \in \text{Irr}(V, \nabla)} G_{I,\mathbf{a}}$ which is compatible with the filtrations \mathcal{F}^Q ($Q \in I$).* ■

We set $\mathcal{F}_a^I := \bigoplus_{\mathbf{b} \leq_I \mathbf{a}} G_{I,\mathbf{b}}$ which is independent of the choice of such a splitting.

Let $I =]\theta_1, \theta_2[$ be as in Lemma 2.22. We choose $\epsilon > 0$ such that $]\theta_1, \theta_1 + \epsilon[$ and $]\theta_2 - \epsilon, \theta_2[$ do not intersect with any of $Z(\mathbf{a}, \mathbf{b})$. Let $\mathcal{H}(\tilde{L}_I)$ denote the space of global sections. Set $Q_1 = \theta_1 + \epsilon$ and $Q_2 = \theta_2 - \epsilon$. We naturally identify it with \tilde{L}_{Q_i} .

Lemma 2.23 *Let $\mathcal{H}(\tilde{L}_I) = \bigoplus_{\mathbf{a} \in \text{Irr}(V, \nabla)} \mathcal{G}_a^{(1)}$ be a decomposition such that $\mathcal{F}_a^{Q_i} \mathcal{H}(\tilde{L}_I) = \bigoplus_{\mathbf{b} \leq_{Q_i} \mathbf{a}} \mathcal{G}_a^{(1)}$. Then, we have $\mathcal{F}_a^Q \mathcal{H}(\tilde{L}_I) = \bigoplus_{\mathbf{b} \leq_Q \mathbf{a}} \mathcal{G}_a^{(1)}$ for any $Q \in I$. In other words, the filtrations \mathcal{F}^Q ($Q \in I$) are determined by \mathcal{F}^{Q_i} ($i = 1, 2$).*

Proof We have a decomposition $\bigoplus \mathcal{G}_a$ as in Lemma 2.22. Set $T(\mathbf{a}, Q_1, Q_2) := \{\mathbf{b} \mid \mathbf{b} \leq_{Q_i} \mathbf{a} (i = 1, 2)\}$. We have

$$\mathcal{G}_a^{(1)} \subset \bigoplus_{\mathbf{b} \in T(\mathbf{a}, Q_1, Q_2)} \mathcal{G}_b$$

Note that for any $\mathbf{b} \in T(\mathbf{a}, Q_1, Q_2)$ and $Q \in I$, we have $\mathbf{b} \leq_Q \mathbf{a}$. Hence, for any $\mathbf{a} \leq_Q \mathbf{c}$, we have $\mathcal{G}_a^{(1)} \subset \mathcal{F}_c^Q$. Then, we can easily deduce the claim of the lemma. ■

2.3.2 Higher dimensional case

Let X be a complex manifold with a simply normal crossing hypersurface H . Let P be any point of H . Let (X_P, z_1, \dots, z_n) be a holomorphic coordinate neighbourhood around P such that $H_P := H \cap X_P = \bigcup_{i=1}^\ell \{z_i = 0\}$. For any $\mathbf{m} \in \mathbb{Z}^\ell$, set $\mathbf{z}^{\mathbf{m}} := \prod_{i=1}^\ell z_i^{m_i}$.

Let $f \in \mathcal{O}_X(*H)_P$. Suppose that for a $\mathbf{m} \in \mathbb{Z}_{\leq 0}^\ell$ the function $\mathbf{z}^{-\mathbf{m}} f$ is holomorphic at P and $(\mathbf{z}^{-\mathbf{m}} f)(P) \neq 0$. Then, we say that f has order \mathbf{m} , and \mathbf{m} is denoted by $\text{ord}(f)$. Note that f does not necessarily have order. We formally set $\text{ord}(0) = (0, \dots, 0)$.

Let $f \in \mathcal{O}_X(*H)_P / \mathcal{O}_{X,P}$. We take a lift $\tilde{f} \in \mathcal{O}_X(*H)_P$ of f . We say that f has order \mathbf{m} if \tilde{f} has order \mathbf{m} . Note that the condition is independent of the choice of a lift \tilde{f} .

Let $\leq_{\mathbb{Z}^\ell}$ denote the partial order on \mathbb{Z}^ℓ given by $\mathbf{m} \leq_{\mathbb{Z}^\ell} \mathbf{m}' \iff m_i \leq m'_i$.

A subset $\mathcal{I} \subset \mathcal{O}_X(*H)_P / \mathcal{O}_{X,P}$ is called a good set of irregular values if the following holds.

- For any $\mathbf{a} \in \mathcal{I}$, we have $\text{ord}(\mathbf{a})$. The set $\{\text{ord}(\mathbf{a}) \mid \mathbf{a}, \mathbf{b} \in \mathcal{I}\}$ is totally ordered with respect to $\leq_{\mathbb{Z}^\ell}$.
- For any $\mathbf{a}, \mathbf{b} \in \mathcal{I}$, we have $\text{ord}(\mathbf{a} - \mathbf{b})$. The set $\{\text{ord}(\mathbf{a} - \mathbf{b}) \mid \mathbf{a}, \mathbf{b} \in \mathcal{I}\}$ is totally ordered with respect to $\leq_{\mathbb{Z}^\ell}$.

A meromorphic flat connection means a coherent reflexive $\mathcal{O}_X(*H)$ -module with an integrable connection (V, ∇) . If V is a locally free $\mathcal{O}_X(*H)$ -module then (V, ∇) is called a meromorphic flat bundle.

Let (V, ∇) be a meromorphic flat bundle on (X, H) . We say that (V, ∇) is unramifiedly good at P if the following holds.

- We have a good set of irregular values $\text{Irr}(V, \nabla, P)$ at P and a decomposition

$$(V, \nabla)|_{\hat{P}} = \bigoplus_{\mathfrak{a} \in \text{Irr}(V, \nabla, P)} (\hat{V}_{\mathfrak{a}}, \hat{\nabla}_{\mathfrak{a}}),$$

such that $\hat{\nabla}_{\mathfrak{a}} - d\tilde{\mathfrak{a}} \text{id}_{\hat{V}_{\mathfrak{a}}}$ are regular singular. Here, $\tilde{\mathfrak{a}} \in \mathcal{O}_X(*H)_P$ are lifts of \mathfrak{a} .

We say that (V, ∇) is unramifiedly good if it is unramifiedly good at any $P \in H$.

We say that (V, ∇) is good at P if the following holds.

- Take a small holomorphic coordinate neighbourhood (X_P, z_1, \dots, z_n) around P such that $H_P := X_P \cap H = \bigcup_{i=1}^{\ell} \{z_i = 0\}$. Let $\varphi_e : \Delta \rightarrow X_P$ given by $\varphi_e(\zeta_1, \dots, \zeta_n) = (\zeta_1^e, \dots, \zeta_{\ell}^e, \zeta_{\ell+1}, \dots, \zeta_n)$. Then there exists e such that $\varphi_e^*(V, \nabla)|_{X_P}$ is unramifiedly good.

We say that (V, ∇) is good if it is good at any point of P .

A meromorphic flat bundle is not necessarily good. The following fundamental theorem is due to Kedlaya [18, 19]. (See [30] for the algebraic case.)

Proposition 2.24 *Let (V, ∇) be any meromorphic flat connection on (X, H) . For any $P \in H$, we have a small neighbourhood X_P and a projective morphism of complex manifolds $\psi_P : \tilde{X}_P \rightarrow X_P$ such that (i) $\tilde{H}_P := \psi_P^{-1}(H)$ is normal crossing, (ii) $\tilde{X}_P \setminus \tilde{H}_P \simeq X_P \setminus H$, (iii) $\psi_P^*(V, \nabla)$ is good.* ■

Let (V, ∇) be an unramifiedly good meromorphic flat connection on (X, H) . Let $\varpi : \tilde{X}(H) \rightarrow X$ be the oriented real blowing up of X along H . Let L be the local system on $X \setminus H$ which is the sheaf of flat sections of $(V, \nabla)|_{X \setminus H}$. Let \tilde{L} be the local system on $\tilde{X}(H)$. Let $Q \in \tilde{X}(H)$. As in the one dimensional case, we have the filtration $\mathcal{F}^Q(\tilde{L}_Q)$ indexed by the ordered set $(\text{Irr}(V, \nabla, \varpi(Q)), \leq_Q)$. The filtration is called the Stokes filtration. We have a splitting $\tilde{L}_Q = \bigoplus G_{Q, \mathfrak{a}}$ such that $\mathcal{F}_{\mathfrak{a}}^Q(\tilde{L}_Q) = \bigoplus_{\mathfrak{b} \leq_Q \mathfrak{a}} G_{Q, \mathfrak{b}}$. If Q' is sufficiently close to Q , we have a natural map $(\text{Irr}(V, \nabla, \varpi(Q)), \leq_Q) \rightarrow (\text{Irr}(V, \nabla, \varpi(Q')), \leq_{Q'})$, which is order preserving. Any splitting of \mathcal{F}^Q induces a splitting of $\mathcal{F}^{Q'}$. The family of Stokes filtrations $\{\mathcal{F}^Q \mid Q \in \varpi^{-1}(H)\}$ is called the Stokes structure associated to (V, ∇) . The following is proved in [30].

Proposition 2.25 *Let \mathcal{I} be a good system of irregular values on (X, H) . We have the functorial equivalence between unramifiedly good meromorphic flat bundles over \mathcal{I} and local systems with Stokes structure over \mathcal{I} .* ■

2.3.3 Extension

Let H be a complex manifold. We set $X = \Delta \times H$. We naturally identify H with $\{0\} \times H$. Let \mathcal{I} be a good system of irregular values on (X, H) . Let H_0 be a complex submanifold of H . We set $X_0 := \Delta \times H_0$. By taking the restriction of \mathcal{I} to H_0 , we obtain a good system of irregular values \mathcal{I}_0 on (X_0, H_0) . The following is a special case of a result in [30].

Proposition 2.26 *Suppose that $H_0 \rightarrow H$ induces an isomorphism of the fundamental groups. We have a functorial bijective correspondence between local systems on $\tilde{X}(H)$ with Stokes structure over \mathcal{I} and local systems on $\tilde{X}_0(H_0)$ with Stokes structure over \mathcal{I}_0 .* ■

Let us observe a variant. Let $X = \Delta^2$, $H_i = \{z_i = 0\}$ and $H = H_1 \cup H_2$. Let $\text{Hol}(X)$ denote the space of holomorphic functions on X . Let $\text{Mero}(X, H)$ be the space of meromorphic functions on (X, H) . Similarly, $\text{Mero}(X, H_i)$ be the space of meromorphic functions.

Let \mathcal{I} be a good set of irregular values on (X, H) . We have $\mathcal{I} \subset \text{Mero}(X, H)/\text{Hol}(X)$. By exchanging z_1 and z_2 if necessary, we may assume the following.

- $\mathcal{I} \rightarrow \text{Mero}(X, H)/\text{Mero}(X, H_2)$ is injective.

Let $\text{MFV}(X, H, \mathcal{I})$ be the category of unramifiedly good meromorphic flat bundles on (X, H, \mathcal{I}) . Let \mathcal{I}_1 be the good system of irregular values on $(X \setminus H_2, H_1 \setminus H_2)$ induced by \mathcal{I} . Let $\text{MFV}(X \setminus H_2, H_1 \setminus H_2, \mathcal{I}_1)$ be the category of unramifiedly good meromorphic flat bundles on $(X \setminus H_2, H_1 \setminus H_2, \mathcal{I}_1)$.

Proposition 2.27 *The restriction induces an equivalence of the categories $\text{MFV}(X, H, \mathcal{I}) \longrightarrow \text{MFV}(X \setminus H_2, H_1 \setminus H_2, \mathcal{I}_1)$.*

Proof We set $X^\circ := X \setminus H_2$ and $H_1^\circ := H_1 \setminus H_2$. Let \tilde{L} be a local system on $\tilde{X}(H)$. Suppose that we are given a Stokes structure \mathcal{F} of $\tilde{L}|_{\tilde{X}^\circ(H_1^\circ)}$ over \mathcal{I}_1 . For any $\mathbf{a}, \mathbf{b} \in \mathcal{I}$ with $\mathbf{a} \neq \mathbf{b}$, we set $F_{\mathbf{a}, \mathbf{b}} := |z^{-m}| \text{Re}(\mathbf{a} - \mathbf{b})$ which naturally gives a C^∞ -function on $\tilde{X}(H)$. Let $Z(\mathbf{a}, \mathbf{b}) = F_{\mathbf{a}, \mathbf{b}}^{-1}(0)$. We use the polar coordinate $z_i = r_i \exp(\sqrt{-1}\theta_i)$. Take $Q = (0, \theta_1^{(0)}, 0, \theta_2^{(0)}) \in \varpi^{-1}(0, 0)$. We take an interval $[\theta_1^{(1)}, \theta_1^{(2)}]$ such that $\theta_1^{(1)} < \theta_1^{(0)} < \theta_1^{(2)}$ and

$$\{(0, \theta_1, 0, \theta_2^{(0)}) \mid \theta_1^{(1)} \leq \theta_1 \leq \theta_1^{(2)}\} \cap \left(\bigcup_{\substack{\mathbf{a}, \mathbf{b} \in \mathcal{I} \\ \mathbf{a} \neq \mathbf{b}}} Z(\mathbf{a}, \mathbf{b}) \right) \subset \{Q\}.$$

We take an interval $[\theta_2^{(1)}, \theta_2^{(2)}]$ such that $\theta_2^{(1)} < \theta_2^{(0)} < \theta_2^{(2)}$ and

$$\{(0, \theta_1^{(i)}, 0, \theta_2) \mid \theta_2^{(1)} \leq \theta_2 \leq \theta_2^{(2)}\} \cap \left(\bigcup_{\substack{\mathbf{a}, \mathbf{b} \in \mathcal{I} \\ \mathbf{a} \neq \mathbf{b}}} Z(\mathbf{a}, \mathbf{b}) \right) = \emptyset \quad (i = 1, 2)$$

We take $\delta > 0$ and $\epsilon > 0$ such that the following holds for $i = 1, 2$:

$$\{(0, \theta_1, r_2, \theta_2) \mid 0 \leq r_2 \leq \delta, \theta_2^{(1)} \leq \theta_2 \leq \theta_2^{(2)}, (\theta_1^{(1)} \leq \theta_1 \leq \theta_1^{(1)} + \epsilon, \text{ or } \theta_1^{(2)} - \epsilon \leq \theta_1 \leq \theta_1^{(2)})\} \cap \left(\bigcup_{\substack{\mathbf{a}, \mathbf{b} \in \mathcal{I} \\ \mathbf{a} \neq \mathbf{b}}} Z(\mathbf{a}, \mathbf{b}) \right) = \emptyset$$

For $0 < r_2 \leq \delta$ and $\theta_2^{(1)} \leq \theta_2 \leq \theta_2^{(2)}$, set $Q_1(r_2, \theta_2) = (0, \theta_1^{(1)} + \epsilon, r_2, \theta_2)$ and $Q_2(r_2, \theta_2) = (0, \theta_1^{(2)} - \epsilon, r_2, \theta_2)$. Let \mathcal{H} denote the space of sections of $\tilde{\mathcal{L}}$ on

$$W = \{(0, \theta_1, r_2, \theta_2) \mid 0 < r_2 \leq \delta, \theta_1^{(1)} \leq \theta_1 \leq \theta_1^{(2)}, \theta_2^{(1)} \leq \theta_2 \leq \theta_2^{(2)}\}.$$

We have the natural isomorphism between \mathcal{H} and $\tilde{\mathcal{L}}_Q$ for any $Q \in W$, with which we identify them. We have the Stokes filtrations $\mathcal{F}^{Q_i(r_2, \theta_2)}$ ($i = 1, 2$) on \mathcal{H} . We can take a decomposition

$$\mathcal{H} = \bigoplus_{\mathbf{a} \in \mathcal{I}} \mathcal{G}_{\mathbf{a}} \tag{3}$$

which gives a splitting to each filtration $\mathcal{F}^{Q_i(r_2, \theta_2)}$, independently from (r_2, θ_2) . By Lemma 2.23, the decomposition (3) gives a splitting of each filtration $\mathcal{F}^{Q'}$ ($Q' \in W$).

Note that $\mathbf{a} \leq_Q \mathbf{b}$ if and only if $\mathbf{a} \leq_{Q'} \mathbf{b}$ for any $Q' \in W$. We define \mathcal{F}^Q by

$$\mathcal{F}_{\mathbf{a}}^Q = \bigoplus_{\mathbf{b} \leq_Q \mathbf{a}} \mathcal{G}_{\mathbf{b}}$$

It is independent of the choice of a decomposition (3). It is easy to see that the family $\{\mathcal{F}^Q \mid Q \in \varpi^{-1}(0, 0)\}$ gives a Stokes structure of $\tilde{\mathcal{L}}$ along $\varpi^{-1}(0, 0)$. Hence, we have a meromorphic flat bundle (V_1, ∇) on a neighbourhood of $(0, 0)$ whose Stokes structure along $\varpi^{-1}(0, 0)$ is given by $\{\mathcal{F}^Q \mid Q \in \varpi^{-1}(0, 0)\}$. It is easy to see that the Stokes filtrations of (V_1, ∇) at $Q' \in \varpi^{-1}(H_1 \setminus H_2)$ are the same as $\mathcal{F}^{Q'}$ for (V, ∇) . Hence, we obtain a Stokes structure of $\tilde{\mathcal{L}}$ whose restriction to $X \setminus H_2$ is equal to the given one. The claim of the proposition follows. \blacksquare

3 Enhanced ind-sheaves

3.1 Preliminary

3.1.1 Ind-sheaves

Let M be any good topological space. Let $H \in IC_M$. For any relatively compact open subset V , we have the natural morphisms $\mathbb{C}_V \otimes H \longrightarrow H$.

Lemma 3.1 *We have $H \simeq \varinjlim_V \mathbb{C}_V \otimes H$ in IC_M .*

Proof We have a small filtrant category I and a functor $\alpha : I \rightarrow \text{Mod}_c(\mathbb{C}_M)$ so that $H = “\varinjlim”\alpha$. We have $\mathbb{C}_V \otimes H = “\varinjlim”(\mathbb{C}_V \otimes \alpha)$ by definition [15]. For any $G \in \text{Mod}_c(\mathbb{C}_M)$, if V is sufficiently large, the induced morphisms $\text{Hom}(G, \mathbb{C}_V \otimes \alpha_i) \rightarrow \text{Hom}(G, \alpha_i)$ are isomorphisms for any $i \in I$. Hence, $\varinjlim_i \text{Hom}(G, \mathbb{C}_V \otimes \alpha_i) \rightarrow \varinjlim_i \text{Hom}(G, \alpha_i)$ is an isomorphism. We regard $\mathbb{C}_V \otimes H$ and H as objects in $\text{Mod}_c(\mathbb{C}_M)^\wedge$. By the above, for any $G \in \text{Mod}_c(\mathbb{C}_M)$, if V is sufficient large, we have $(\mathbb{C}_V \otimes H)(G) = H(G)$. It implies that $\varinjlim_V \mathbb{C}_V \otimes H \rightarrow H$ is an isomorphism in $\text{Mod}_c(\mathbb{C}_M)^\wedge$. Because $IC_M \rightarrow \text{Mod}_c(\mathbb{C}_M)^\wedge$ commute with inductive limits [16, Theorem 6.1.8], we obtain $\varinjlim_V \mathbb{C}_V \otimes H \simeq H$ in IC_M . \blacksquare

Let U be any locally closed relatively compact subset of M .

Lemma 3.2 *The functor $H \mapsto \mathbb{C}_U \otimes H$ gives an exact functor on IC_M .*

Proof Let $0 \rightarrow H^{(1)} \rightarrow H^{(2)} \rightarrow H^{(3)} \rightarrow 0$ be an exact sequence in IC_M . We have a filtrant small category I and an exact sequence of functors $0 \rightarrow \alpha^{(1)} \rightarrow \alpha^{(2)} \rightarrow \alpha^{(3)} \rightarrow 0$ from I to $\text{Mod}_c(\mathbb{C}_M)$ such that $0 \rightarrow “\varinjlim”\alpha^{(1)} \rightarrow “\varinjlim”\alpha^{(2)} \rightarrow “\varinjlim”\alpha^{(3)} \rightarrow 0$ is $0 \rightarrow H^{(1)} \rightarrow H^{(2)} \rightarrow H^{(3)} \rightarrow 0$. (See [15, Theorem 1.3.1] and the remark right after the theorem.) By [14, Proposition 2.3.6, Proposition 2.3.10], $0 \rightarrow \mathbb{C}_U \otimes \alpha_i^{(1)} \rightarrow \mathbb{C}_U \otimes \alpha_i^{(2)} \rightarrow \mathbb{C}_U \otimes \alpha_i^{(3)} \rightarrow 0$ are exact for any $i \in I$, we obtain that $0 \rightarrow \mathbb{C}_U \otimes H^{(1)} \rightarrow \mathbb{C}_U \otimes H^{(2)} \rightarrow \mathbb{C}_U \otimes H^{(3)} \rightarrow 0$ is exact. \blacksquare

Let Z be a closed subset of U . We set $U_0 := U \setminus Z$. We have the exact sequence $0 \rightarrow \mathbb{C}_{U_0} \rightarrow \mathbb{C}_U \rightarrow \mathbb{C}_Z \rightarrow 0$ in $\text{Mod}_c(\mathbb{C}_M)$. For any $H \in IC_M$, we obtain the following morphisms in IC_M :

$$0 \rightarrow \mathbb{C}_{U_0} \otimes H \rightarrow \mathbb{C}_U \otimes H \rightarrow \mathbb{C}_Z \otimes H \rightarrow 0 \quad (4)$$

Lemma 3.3 *(4) is exact.*

Proof We have an expression $H = “\varinjlim”\alpha$ for a functor α from a small filtrant category I to $\text{Mod}_c(\mathbb{C}_M)$. The sequences $0 \rightarrow \mathbb{C}_{U_0} \otimes \alpha_i \rightarrow \mathbb{C}_U \otimes \alpha_i \rightarrow \mathbb{C}_Z \otimes \alpha_i \rightarrow 0$ are exact for any i . Hence, we obtain that (4) is exact. \blacksquare

3.1.2 Enhanced ind-sheaves

Let $\overline{\mathbb{R}} := \mathbb{R} \sqcup \{\infty, -\infty\}$ be the compactification of \mathbb{R} . We have the bordered spaces $\mathbb{R}_\infty = (\mathbb{R}, \overline{\mathbb{R}})$ and $\overline{\mathbb{R}} = (\overline{\mathbb{R}}, \overline{\mathbb{R}})$. Let M be any good topological space. Let $j : M \times \mathbb{R}_\infty \rightarrow M \times \overline{\mathbb{R}}$ be the inclusion. Let $\pi : M \times \mathbb{R}_\infty \rightarrow M$ and $\overline{\pi} : M \times \overline{\mathbb{R}} \rightarrow M$ denote the projections.

Let $\mathcal{H}^0 \mathbf{E}^b(IC_M)$ denote the heart of $\mathbf{E}^b(IC_M)$ with respect to the t -structure in [3, §4.6]. Recall that an object $K \in \mathbf{E}^b(IC_M)$ is contained in $\mathcal{H}^0 \mathbf{E}^b(IC_M)$ if and only if the object $Rj_{!!} L^E K \in D^b(IC_{\overline{\mathbb{R}} \times M})$ is contained in $IC_{\overline{\mathbb{R}} \times M}$. For such K , and for any relatively compact open subset V of M , we have $\pi^{-1}(\mathbb{C}_V) \otimes K$ in $\mathcal{H}^0 \mathbf{E}^b(IC_M)$. Note that $Rj_{!!} L^E(\pi^{-1}(\mathbb{C}_V) \otimes K) \simeq \overline{\pi}^{-1}(\mathbb{C}_V) \otimes Rj_{!!} L^E(K)$. For inclusions $V_1 \subset V_2$, we have the natural morphisms $\pi^{-1}(\mathbb{C}_{V_1}) \otimes K \rightarrow \pi^{-1}(\mathbb{C}_{V_2}) \otimes K$.

Lemma 3.4 *We have the natural isomorphism $K \simeq \varinjlim_V \pi^{-1}(\mathbb{C}_V) \otimes K$ in $\mathcal{H}^0 \mathbf{E}^b(IC_M)$.*

Proof By the definition of the t -structures, we have the fully faithful embedding of the abelian categories $\mathcal{H}^0 \mathbf{E}^b(IC_M) \rightarrow IC_{\overline{\mathbb{R}} \times M}$ induced by $G \mapsto Rj_{!!} L^E(G)$. By Lemma 3.1, we have $Rj_{!!} L^E(K) \simeq \varinjlim_V \overline{\pi}^{-1}(\mathbb{C}_V) \otimes Rj_{!!} L^E(K) \simeq \varinjlim_V Rj_{!!} L^E(\pi^{-1}(\mathbb{C}_V) \otimes K)$. Hence, we have $K \simeq \varinjlim_V \pi^{-1}(\mathbb{C}_V) \otimes K$ in $\mathcal{H}^0 \mathbf{E}^b(IC_M)$. \blacksquare

Let U be any relatively compact locally closed subset of M . We obtain the following from Lemma 3.2

Lemma 3.5 *The functor $K \mapsto \pi^{-1}(\mathbb{C}_U) \otimes K$ on $\mathbf{E}^b(\mathbb{C}_M)$ induces an exact functor on $\mathcal{H}^0 \mathbf{E}^b(\mathbb{C}_M)$.* \blacksquare

Let Z be a closed subset in U . Set $U_0 := U \setminus Z$. We obtain the following from Lemma 3.3.

Lemma 3.6 *$0 \rightarrow \overline{\pi}^{-1}(\mathbb{C}_{U_0}) \otimes K \rightarrow \overline{\pi}^{-1}(\mathbb{C}_U) \otimes K \rightarrow \overline{\pi}^{-1}(\mathbb{C}_Z) \otimes K \rightarrow 0$ is an exact sequence in $\mathcal{H}^0 \mathbf{E}^b(IC_M)$.* \blacksquare

3.2 Stable free enhanced ind-sheaves

3.2.1 Enhanced ind-sheaves associated to global subanalytic functions

Suppose that M is a subanalytic space. Let $U \subset M$ be any locally closed subanalytic subset. Let g be a continuous subanalytic function on (U, M) , i.e., it is a continuous function on U , and the graph of g is subanalytic in $\mathbb{P}^1(\mathbb{R}) \times M$. Set $Z(g) := \{(x, t) \in U \times \mathbb{R} \mid t \geq g(x)\} \subset M \times \mathbb{R}$. Let $\mathbb{C}_{t \geq g}$ denote the \mathbb{R} -constructible sheaf $\mathbb{C}_{Z(g)}$ on $M \times \mathbb{R}$, which can be extended to an \mathbb{R} -constructible sheaf on $M \times \overline{\mathbb{R}}$. We have $\mathbb{C}_M^E \overset{+}{\otimes} \mathbb{C}_{t \geq g}$ in $\mathcal{H}^0 \mathbf{E}^b(IC_M)$. (See [3].)

Lemma 3.7 *Let L be any local system on M . Let g_i ($i = 1, 2$) be continuous subanalytic functions on M such that $g_1(x) \leq g_2(x)$ for any $x \in M$. The natural morphism $\kappa : \pi^{-1}(L) \otimes (\mathbb{C}_M^E \overset{+}{\otimes} \mathbb{C}_{t \geq g_1}) \longrightarrow \pi^{-1}(L) \otimes (\mathbb{C}_M^E \overset{+}{\otimes} \mathbb{C}_{t \geq g_2})$ is an isomorphism in $\mathbf{E}^b(IC_M)$.*

Proof By Lemma 3.4, it is enough to prove that the natural morphism

$$\pi^{-1}(\mathbb{C}_V \otimes L) \otimes (\mathbb{C}_M^E \overset{+}{\otimes} \mathbb{C}_{t \geq g_1}) \longrightarrow \pi^{-1}(\mathbb{C}_V \otimes L) \otimes (\mathbb{C}_M^E \overset{+}{\otimes} \mathbb{C}_{t \geq g_2}) \quad (5)$$

is an isomorphism for any relatively compact open subanalytic subset $V \subset M$. By [3, Lemma 4.3.1], we naturally have

$$\pi^{-1}(\mathbb{C}_V \otimes L) \otimes (\mathbb{C}_M^E \overset{+}{\otimes} \mathbb{C}_{t \geq g_i}) \simeq \mathbb{C}_M^E \overset{+}{\otimes} (\pi^{-1}(L|_V) \otimes \mathbb{C}_{t \geq g_i|_V}).$$

By [3, Proposition 4.7.9], we have

$$\begin{aligned} \mathrm{Hom}_{\mathbf{E}^b(IC_M)} \left(\mathbb{C}_M^E \overset{+}{\otimes} (\pi^{-1}(L|_V) \otimes \mathbb{C}_{t \geq g_i|_V}), \mathbb{C}_M^E \overset{+}{\otimes} (\pi^{-1}(L|_V) \otimes \mathbb{C}_{t \geq g_j|_V}) \right) \\ \simeq \varinjlim_{a \rightarrow \infty} \mathrm{Hom}_{\mathrm{Mod}(\mathbb{C}_{M \times \mathbb{R}})} (\pi^{-1}(L|_V) \otimes \mathbb{C}_{t \geq g_i|_V}, \pi^{-1}(L|_V) \otimes \mathbb{C}_{t \geq g_j|_V + a}). \end{aligned} \quad (6)$$

We take a large number b such that $g_2(x) \leq g_1(x) + b$ for any $x \in V$. We have the natural morphism $\pi^{-1}(L|_V) \otimes \mathbb{C}_{t \geq g_2|_V} \longrightarrow \pi^{-1}(L|_V) \otimes \mathbb{C}_{t \geq g_1|_V + b}$. By (6), it induces a morphism

$$\kappa_1 : \mathbb{C}_M^E \overset{+}{\otimes} (\pi^{-1}(L|_V) \otimes \mathbb{C}_{t \geq g_2|_V}) \longrightarrow \mathbb{C}_M^E \overset{+}{\otimes} (\pi^{-1}(L|_V) \otimes \mathbb{C}_{t \geq g_1|_V})$$

in $\mathbf{E}^b(IC_M)$. The composite $\kappa_1 \circ \kappa$ is induced by the natural morphism $\pi^{-1}(L|_V) \otimes \mathbb{C}_{t \geq g_1|_V} \longrightarrow \pi^{-1}(L|_V) \otimes \mathbb{C}_{t \geq g_1|_V + b}$. Hence, we can easily observe that $\kappa_1 \circ \kappa$ is equal to the identity of $\mathbb{C}_M^E \overset{+}{\otimes} \pi^{-1}(L|_V) \otimes \mathbb{C}_{t \geq g_1|_V}$. Similarly, we can observe that $\kappa \circ \kappa_1$ is the identity. Hence, the morphism (5) is an isomorphism. \blacksquare

For any continuous subanalytic function g on M , we define the continuous subanalytic function g_- on M by $g_-(x) := \min\{0, g(x)\}$. We have the following natural morphisms

$$\pi^{-1}(L) \otimes \mathbb{C}_M^E \simeq \pi^{-1}(L) \otimes (\mathbb{C}_M^E \overset{+}{\otimes} \mathbb{C}_{t \geq 0}) \xleftarrow{c_1} \pi^{-1}(L) \otimes (\mathbb{C}_M^E \overset{+}{\otimes} \mathbb{C}_{t \geq g_-}) \xrightarrow{c_2} \pi^{-1}(L) \otimes (\mathbb{C}_M^E \overset{+}{\otimes} \mathbb{C}_{t \geq g}).$$

Corollary 3.8 *The morphisms c_i are isomorphisms in $\mathbf{E}^b(IC_M)$. In this sense, we have a canonical isomorphism $\pi^{-1}(L) \otimes \mathbb{C}_M^E \simeq \pi^{-1}(L) \otimes (\mathbb{C}_M^E \overset{+}{\otimes} \mathbb{C}_{t \geq g})$ for any continuous subanalytic function g on M .* \blacksquare

3.2.2 Pre-orders on the sets of continuous subanalytic functions

Let M be a real analytic manifold which may have a boundary. Let U be a subanalytic locally closed subset in M . Let $\mathrm{Sub}(U, M)$ denote the set of continuous subanalytic functions on (U, M) . We have the pre-order \prec on $\mathrm{Sub}(U, M)$ for which we have $g \prec f$ if and only if $f - g$ is bounded from above on $V \cap U$ for any relatively compact subset V in M . We have the equivalence relation \sim on $\mathrm{Sub}(U, M)$ defined by $g \sim f \stackrel{\mathrm{def}}{\iff} g \prec f$ and $f \prec g$. Let $\overline{\mathrm{Sub}}(U, M)$ denote the quotient set of $\mathrm{Sub}(U, M)$ by the above equivalence relation \sim .

Let U' be a locally closed subanalytic subset in a real analytic manifold M' . Let $\varphi : M' \longrightarrow M$ be an analytic map such that $\varphi(U') \subset U$. We have the map $\mathrm{Sub}(U, M) \longrightarrow \mathrm{Sub}(U', M')$ given by the pull back $f \mapsto \varphi^*(f)$. If $f \prec g$ in $\mathrm{Sub}(U, M)$, then $\varphi^*f \prec \varphi^*g$ in $\mathrm{Sub}(U', M')$. We have the induced map $\overline{\mathrm{Sub}}(U, M) \longrightarrow \overline{\mathrm{Sub}}(U', M')$.

Lemma 3.9 For $\psi_i \in \text{Sub}(U, M)$ ($i = 1, 2$), we have $\psi_1 \prec \psi_2$ if and only if the following holds.

- We set $I :=]0, 1]$ and $I^\circ := I \setminus \{0\}$. Let $\varphi : I \rightarrow M$ be any real analytic map such that $\varphi(I^\circ) \subset U$. Then, we have $\varphi^*(\psi_1) \prec \varphi^*(\psi_2)$ in $\text{Sub}(I^\circ, I)$.

Proof The “only if” part is clear. Let us prove the “if” part. Suppose that $\psi_1 \not\prec \psi_2$. Set $h := \psi_1 - \psi_2$. We have a relatively compact subset $V \subset M$ such that h is not bounded from above on $V \cap U$. Let $\Gamma_{h, V \cap U}$ denote the graph of $h|_{V \cap U}$. The closure of $\Gamma_{h, V \cap U}$ in $\mathbb{R}_\infty \times M$, which contains a point in $\{\infty\} \times M$. By the curve selection lemma, we can choose a real analytic map $\varphi : I \rightarrow M$ such that $\varphi(I^\circ) \subset U$ and $\varphi^*(h)$ is not bounded from above, i.e., $\varphi^*(\psi_1) \not\prec \varphi^*(\psi_2)$. \blacksquare

3.2.3 Spaces of morphisms in some basic cases

Let $U \subset M$ be a locally closed subanalytic subset. Let $\iota_U : U \rightarrow M$ denote the inclusion. Let L_i ($i = 1, 2$) be a local system on U . We put $L_{i, M} := \iota_{U!} L_i \in \text{Mod}(\mathbb{C}_M)$. Let g_i ($i = 1, 2$) be continuous subanalytic functions on (U, M) . By Corollary 3.8, we have $\text{E}\iota_U^{-1} \left(\mathbb{C}_M^{\text{E}} \otimes^+ (\pi^{-1}(L_{i, M}) \otimes \mathbb{C}_{t \geq g_i}) \right) \simeq \mathbb{C}_U^{\text{E}} \otimes \pi^{-1}(L_i)$ in $\text{E}^b(IC_U)$. Recall that we have the fully faithful embedding $D^b(\mathbb{C}_U) \rightarrow \text{E}^b(IC_U)$ given by $G \mapsto \pi^{-1}(G) \otimes \mathbb{C}_M^{\text{E}}$. (See [15, Proposition 5.1.1] and [3, Proposition 4.7.15].) Hence, the functor $\text{E}\iota_U^{-1}$ induces the following morphism:

$$\text{Hom}_{\text{E}^b(IC_M)} \left(\mathbb{C}_M^{\text{E}} \otimes^+ (\pi^{-1}(L_{1, M}) \otimes \mathbb{C}_{t \geq g_1}), \mathbb{C}_M^{\text{E}} \otimes^+ (\pi^{-1}(L_{2, M}) \otimes \mathbb{C}_{t \geq g_2}) \right) \rightarrow \text{Hom}_{\text{Mod}(\mathbb{C}_U)}(L_1, L_2). \quad (7)$$

Lemma 3.10 The morphism (7) is injective. The morphism is an isomorphism if $g_2 \prec g_1$ in $\text{Sub}(U, M)$. If U is connected and if $g_2 \not\prec g_1$ in $\text{Sub}(U, M)$, then we have

$$\text{Hom}_{\text{E}^b(IC_M)} \left(\mathbb{C}_M^{\text{E}} \otimes^+ (\pi^{-1}(L_{1, M}) \otimes \mathbb{C}_{t \geq g_1}), \mathbb{C}_M^{\text{E}} \otimes^+ (\pi^{-1}(L_{2, M}) \otimes \mathbb{C}_{t \geq g_2}) \right) = 0.$$

Proof Let V be any relatively compact open subanalytic subset in M . We have the following morphisms:

$$\begin{aligned} & \text{Hom}_{\text{E}^b(IC_M)} \left(\mathbb{C}_M^{\text{E}} \otimes^+ (\pi^{-1}(\mathbb{C}_V \otimes L_{1, M}) \otimes \mathbb{C}_{t \geq g_1}), \mathbb{C}_M^{\text{E}} \otimes^+ (\pi^{-1}(L_{2, M}) \otimes \mathbb{C}_{t \geq g_2}) \right) \\ & \simeq \varinjlim_{a \rightarrow \infty} \text{Hom}_{\text{Mod}(\mathbb{C}_{M \times \mathbb{R}})} (\pi^{-1}(\mathbb{C}_V \otimes L_{1, M}) \otimes \mathbb{C}_{t \geq g_1 - a}, \pi^{-1}(L_{2, M}) \otimes \mathbb{C}_{t \geq g_2}) \\ & \xrightarrow{\kappa_V} \text{Hom}_{\text{Mod}(\mathbb{C}_{M \times \mathbb{R}})} (\pi^{-1}(\mathbb{C}_V \otimes L_{1, M}), \pi^{-1}(L_{2, M}) \otimes \mathbb{C}_{t \geq g_2}) \\ & \simeq \text{Hom}_{\text{Mod}(\mathbb{C}_{M \times \mathbb{R}})} (\pi^{-1}(\mathbb{C}_V \otimes L_{1, M}), \pi^{-1}(L_{2, M})) \simeq \text{Hom}_{\mathbb{C}_U} (\mathbb{C}_{U \cap V} \otimes L_1, L_2) \end{aligned} \quad (8)$$

Here, κ_V is injective. Moreover, κ_V is an isomorphism if $g_1 - g_2$ is bounded from above on $U \cap V$. The morphism (7) is equal to the projective limit of (8), where V runs through the set of relatively compact open subsets of M . Hence, we obtain that (7) is injective, and that (7) is an isomorphism if $g_2 \prec g_1$.

Suppose that U is connected and $g_2 \not\prec g_1$. We have $P \in \overline{U}$ around which $g_1 - g_2$ is not bounded from above. We take a relatively compact subanalytic open neighbourhood V around P . We have the decomposition $V \cap U = \coprod \mathcal{C}_i$ into connected components. We have i_0 such that $g_1 - g_2$ is not bounded from above on \mathcal{C}_{i_0} . We have the following:

$$\begin{aligned} & \text{Hom}_{\text{E}^b(IC_M)} \left(\mathbb{C}_M^{\text{E}} \otimes^+ (\pi^{-1}(\mathbb{C}_{\mathcal{C}_{i_0}} \otimes L_{1, M}) \otimes \mathbb{C}_{t \geq g_1}), \mathbb{C}_M^{\text{E}} \otimes^+ (\pi^{-1}(L_{2, M}) \otimes \mathbb{C}_{t \geq g_2}) \right) \\ & \simeq \varinjlim_{a \rightarrow \infty} \text{Hom}_{\text{Mod}(\mathbb{C}_{M \times \mathbb{R}})} (\pi^{-1}(\mathbb{C}_{\mathcal{C}_{i_0}} \otimes L_{1, M}) \otimes \mathbb{C}_{t \geq g_1 - a}, \pi^{-1}(L_{2, M}) \otimes \mathbb{C}_{t \geq g_2}) = 0 \end{aligned} \quad (9)$$

Let s be an element of the image of (7). By (9), the restriction of s to \mathcal{C}_{i_0} is 0. Because U is assumed to be connected, we obtain $s = 0$. \blacksquare

Corollary 3.11 Suppose that $g_1 - g_2$ is positive and that $g_2 \not\prec g_1$. Then, $\mathbb{C}_M^{\text{E}} \otimes^+ \mathbb{C}_{g_2 \leq t < g_1}$ is not isomorphic to 0 in $\text{E}^b(IC_M)$.

Proof We have the distinguished triangle $\mathbb{C}_M^E \overset{+}{\otimes} \mathbb{C}_{g_2 \leq t < g_1} \longrightarrow \mathbb{C}_M^E \overset{+}{\otimes} \mathbb{C}_{t \geq g_2} \xrightarrow{a} \mathbb{C}_M^E \overset{+}{\otimes} \mathbb{C}_{t \geq g_1} \longrightarrow \mathbb{C}_M^E \overset{+}{\otimes} \mathbb{C}_{g_2 \leq t < g_1}[1]$. Because a is not an isomorphism in $E^b(IC_M)$ by Lemma 3.10, we have $\mathbb{C}_M^E \overset{+}{\otimes} \mathbb{C}_{g_2 \leq t < g_1} \neq 0$. \blacksquare

Let $\iota_U : U \longrightarrow M$ be the inclusion. The restriction $E\iota_U^{-1}(K)$ is denoted by $K|_U$. If $K = \mathbb{C}_M^E \overset{+}{\otimes} \mathbb{C}_{t \geq g_i}$, $K|_U$ is obtained from the constant sheaf \mathbb{C}_U on U .

Corollary 3.12 *Suppose that we are given an isomorphism $\Phi : (\mathbb{C}_M^E \overset{+}{\otimes} \mathbb{C}_{t \geq g_1})|_U \simeq (\mathbb{C}_M^E \overset{+}{\otimes} \mathbb{C}_{t \geq g_2})|_U$. It is prolonged to an isomorphism $\tilde{\Phi} : \mathbb{C}_M^E \overset{+}{\otimes} \mathbb{C}_{t \geq g_1} \longrightarrow \mathbb{C}_M^E \overset{+}{\otimes} \mathbb{C}_{t \geq g_2}$ if and only if the following holds:*

- Set $I = [0, 1[$ and $I^\circ := I \setminus \{0\}$. Let $\varphi : I \longrightarrow M$ be any real analytic map such that $\varphi(I^\circ) \subset U$. Then, the isomorphism $\varphi^{-1}(\Phi)$ is extended to an isomorphism $E\varphi^{-1}(\mathbb{C}_M^E \overset{+}{\otimes} \mathbb{C}_{t \geq g_1}) \longrightarrow E\varphi^{-1}(\mathbb{C}_M^E \overset{+}{\otimes} \mathbb{C}_{t \geq g_2})$.

Such an isomorphism $\tilde{\Phi}$ is unique if it exists.

Proof The “only if” part is clear. Let us see the “if” part. Under the assumption, we have $\varphi^{-1}(g_2) \prec \varphi^{-1}(g_1)$ for any real analytic map $\varphi : I \longrightarrow M$ such that $\varphi(I^\circ) \subset U$. Hence, we have $g_2 \prec g_1$ by Lemma 3.9. \blacksquare

Lemma 3.13 *Let h, φ and ψ be continuous subanalytic functions on (U, M) . We assume $\varphi(P) < \psi(P)$ for any $P \in U$. Let L_i ($i = 1, 2$) be local systems on U . We have*

$$\mathrm{Hom}_{E^b(IC_M)}\left(\mathbb{C}_M^E \overset{+}{\otimes} (\pi^{-1}(L_{1,M}) \otimes \mathbb{C}_{t \geq h}), \mathbb{C}_M^E \overset{+}{\otimes} (\pi^{-1}(L_{2,M}) \otimes \mathbb{C}_{\varphi \leq t < \psi})\right) = 0. \quad (10)$$

Proof Let V be a relatively compact subanalytic open subset in M . We have

$$\begin{aligned} \mathrm{Hom}_{E^b(IC_M)}\left(\mathbb{C}_M^E \overset{+}{\otimes} (\pi^{-1}(\mathbb{C}_V \otimes L_{1,M}) \otimes \mathbb{C}_{t \geq h}), \mathbb{C}_M^E \overset{+}{\otimes} (\mathbb{C}_{\varphi \leq t < \psi} \otimes \pi^{-1}(L_{2,M}))\right) &\simeq \\ \lim_{a \rightarrow \infty} \mathrm{Hom}_{\mathrm{Mod}(\mathbb{C}_M \times \mathbb{R})}\left((\pi^{-1}(\mathbb{C}_V \otimes L_{1,M}) \otimes \mathbb{C}_{t \geq h-a}), \pi^{-1}(L_{2,M}) \otimes \mathbb{C}_{\varphi \leq t < \psi}\right) &= 0. \end{aligned} \quad (11)$$

Then, we obtain (10) by taking the projective limit. \blacksquare

Similarly, we have the following.

Lemma 3.14 *Let φ_i and ψ_i ($i = 1, 2$) be continuous subanalytic functions on (U, M) . We assume $\varphi_i(P) < \psi_i(P)$ for any $P \in U$. Let L_i ($i = 1, 2$) be local systems on U . Unless $\varphi_2 \prec \varphi_1$ and $\psi_2 \prec \psi_1$, we have*

$$\mathrm{Hom}_{E^b(IC_M)}\left(\mathbb{C}_M^E \overset{+}{\otimes} (\pi^{-1}(L_{1,M}) \otimes \mathbb{C}_{\varphi_1 \leq t < \psi_1}), \mathbb{C}_M^E \overset{+}{\otimes} (\pi^{-1}(L_{2,M}) \otimes \mathbb{C}_{\varphi_2 \leq t < \psi_2})\right) = 0. \quad (12)$$

3.2.4 Canonical filtrations

Let U be a connected subanalytic locally closed subset in M . Let $I = \{g_1, \dots, g_m\}$ be a finite tuple in $\mathrm{Sub}(U, M)$. Let \overline{I} denote the image of $I \longrightarrow \overline{\mathrm{Sub}}(U, M)$. We have the induced order \prec on \overline{I} . For each $g_j \in I$, let $[g_j]$ denote the induced element in \overline{I} . We consider an object of the form $K = \bigoplus_{i=1}^m \mathbb{C}_M^E \overset{+}{\otimes} \mathbb{C}_{t \geq g_i}$ in $E^b(IC_M)$. It is equipped with the filtration \mathcal{F}^U indexed by (\overline{I}, \prec) given as follows:

$$\mathcal{F}_{[g]}^U(K) := \bigoplus_{\substack{g_i \in I \\ [g_i] \prec [g]}} \mathbb{C}_M^E \overset{+}{\otimes} \mathbb{C}_{t \geq g_i}.$$

We may regard \mathcal{F} as a filtration indexed by $(\overline{\mathrm{Sub}}(U, M), \prec_U)$.

Lemma 3.15 Let $I_1 = \{g_1, \dots, g_m\}$ and $I_2 = \{h_1, \dots, h_\ell\}$ be tuples of continuous subanalytic functions on U . Suppose that

$$K = \bigoplus_{i=1}^m \mathbb{C}_M^E \otimes^+ \mathbb{C}_{t \geq g_i} = \bigoplus_{j=1}^\ell \mathbb{C}_M^E \otimes^+ \mathbb{C}_{t \geq h_j}$$

Then, we have $(\bar{I}_1, \prec) = (\bar{I}_2, \prec)$, and the induced filtrations on K are the same.

Proof Let $a \in \bar{I}_1$ be a minimal element. We have a minimal element $b \in \bar{I}_2$ such that $b \prec a$. We also have a minimal element $c \in \bar{I}_1$ such that $c \prec b$. We have $c \prec a$. Because a and c are minimal, we have $a = c = b$. The composite of the morphisms

$$\bigoplus_{j=1}^m \mathbb{C}_M^E \otimes^+ \mathbb{C}_{t \geq g_j} \longrightarrow \bigoplus_{j=1}^\ell \mathbb{C}_M^E \otimes^+ \mathbb{C}_{t \geq h_j} \longrightarrow \mathbb{C}_M^E \otimes^+ \left(\bigoplus_{j=1}^\ell \mathbb{C}_{t \geq h_j} / \bigoplus_{[h_j]=b} \mathbb{C}_{t \geq h_j} \right)$$

induces

$$\mathbb{C}_M^E \otimes^+ \left(\bigoplus_{j=1}^m \mathbb{C}_{t \geq g_j} / \bigoplus_{[g_j]=b} \mathbb{C}_{t \geq g_j} \right) \longrightarrow \mathbb{C}_M^E \otimes^+ \left(\bigoplus_{j=1}^\ell \mathbb{C}_{t \geq h_j} / \bigoplus_{[h_j]=b} \mathbb{C}_{t \geq h_j} \right). \quad (13)$$

It is easy to see that the morphism (13) is an isomorphism. Hence, by an easy induction, we obtain the claim of the lemma. \blacksquare

If $K = \bigoplus_{g \in I} \mathbb{C}_M^E \otimes^+ \mathbb{C}_{t \geq g}$, we say that the tuple I controls the growth order of K . For such K , the restriction $K|_U \in \mathbf{E}^b(I\mathbb{C}_U)$ is isomorphic to $(\mathbb{C}_U^E)^{\oplus |I|}$, i.e., it comes from the free \mathbb{C}_U -module which is equipped with the induced filtration \mathcal{F} indexed by (\bar{I}, \prec) . We can recover K from the free \mathbb{C}_U -module with the filtration \mathcal{F} .

Lemma 3.16 Let I_i ($i = 1, 2$) be tuples of continuous subanalytic functions on (U, M) . We set $K_i := \bigoplus_{g \in I_i} \mathbb{C}_M^E \otimes^+ \mathbb{C}_{t \geq g}$. Let (L_i, \mathcal{F}) be the underlying filtered \mathbb{C}_U -free modules. We have a natural bijection

$$\mathrm{Hom}_{\mathbf{E}^b(I\mathbb{C}_M)}(K_1, K_2) \simeq \{f \in \mathrm{Hom}_{\mathrm{Mod}(\mathbb{C}_U)}(L_1, L_2) \mid f(\mathcal{F}_a L_1) \subset \mathcal{F}_a L_2 \ (\forall a \in \overline{\mathrm{Sub}}(U, M))\}.$$

In particular, we have the natural injection $\mathrm{Hom}_{\mathbf{E}^b(I\mathbb{C}_M)}(K_1, K_2) \longrightarrow \mathrm{Hom}_{\mathrm{Mod}(\mathbb{C}_U)}(L_1, L_2)$.

Proof It follows from Lemma 3.10. \blacksquare

Let I_i ($i = 1, 2$) be tuples of continuous subanalytic functions on (U, M) . Let $\varphi_j < \psi_j$ ($j = 1, \dots, m$) be subanalytic functions on (U, M) . We set

$$K_i := \bigoplus_{g \in I_i} \mathbb{C}_M^E \otimes^+ \mathbb{C}_{t \geq g}, \quad F := \bigoplus_{j=1}^m \mathbb{C}_M^E \otimes^+ \mathbb{C}_{\varphi_j \leq t < \psi_j}.$$

By Lemma 3.13, we have $\mathrm{Hom}_{\mathbf{E}^b(I\mathbb{C}_M)}(K_i, F) = 0$.

Lemma 3.17 If $\rho : K_1 \simeq K_2 \oplus F$ in $\mathbf{E}^b(I\mathbb{C}_M)$, then we have $F = 0$ and $K_1 \simeq K_2$. In particular, we have $\psi_j \sim \varphi_j$ for any j . We also have $\bar{I}_1 = \bar{I}_2$ in $\overline{\mathrm{Sub}}(U, M)$.

Proof We have the natural morphisms $a : F \longrightarrow K_2 \oplus F$ and $b : K_2 \oplus F \longrightarrow F$ for which we have $b \circ a = \mathrm{id}_F$. Because any morphism $K_1 \longrightarrow F$ is 0, we obtain $\mathrm{id}_F = 0$, and hence $F = 0$. \blacksquare

3.2.5 Prolongations of morphisms

Let $I_i \subset \mathrm{Sub}(U, M)$ ($i = 1, 2$) be finite subsets. We set $K_i := \bigoplus_{g \in I_i} \mathbb{C}_M^E \otimes^+ \mathbb{C}_{t \geq g}$. We have the corresponding local systems with a filtration (L_i, \mathcal{F}^U) on U . Suppose that we are given a morphism of local systems $\Phi : L_1 \longrightarrow L_2$ on U . A prolongation of Φ is a morphism $\tilde{\Phi} : K_1 \longrightarrow K_2$ whose restriction to U is equal to Φ . Such a prolongation is unique if it exists.

Lemma 3.18 *We have a prolongation of Φ if and only if the following holds.*

- Set $I = [0, 1[$ and $I^\circ := I \setminus \{0\}$. Let $\varphi : I \rightarrow M$ be any real analytic map such that $\varphi(I^\circ) \subset U$. Then, the morphism $\varphi_{|I^\circ}^{-1}(\Phi) : \varphi_{|I^\circ}^{-1}L_1 \rightarrow \varphi_{|I^\circ}^{-1}L_2$ is prolonged to a morphism $E\varphi^{-1}(K_1) \rightarrow E\varphi^{-1}(K_2)$.

Proof The “only if” part is clear. The “if” part follows from Corollary 3.12. ■

3.2.6 Appendix: Canonical filtrations on stable free enhanced ind-sheaves

We can also consider a canonical filtration in a more general case. We set $\overline{\text{Sub}}^*(U, M) := \overline{\text{Sub}}(U, M) \sqcup \{\infty\}$. We have the order \prec on $\overline{\text{Sub}}^*(U, M)^2$ given by $(a, b) \prec (a', b') \stackrel{\text{def}}{\iff} a \prec a'$ and $b \prec b'$.

We set $\text{Sub}^*(U, M) := \text{Sub}(U, M) \sqcup \{\infty\}$. Let (φ_i, ψ_i) ($i = 1, \dots, m$) be pairs in $\text{Sub}^*(U, M)$ such that $\varphi_i < \psi_i$. We consider any object of the form $K = \bigoplus_{i=1}^m \mathbb{C}_M^E \overset{+}{\otimes} \mathbb{C}_{\varphi_i \leq t < \psi_i}$ in $E^b(IC_M)$. Then, we have the filtration \mathcal{F}^U on K indexed by $(\overline{\text{Sub}}^*(U, M)^2, \prec)$ given as follows:

$$\mathcal{F}_{(\varphi, \psi)}^U(K) := \bigoplus_{([\varphi_i], [\psi_i]) \prec ([\varphi], [\psi])} \mathbb{C}_M^E \overset{+}{\otimes} \mathbb{C}_{\varphi_i \leq t < \psi_i}.$$

It is canonically defined for K , which can be shown as in the case of Lemma 3.15.

3.3 Prolongations of local systems

3.3.1 Prolongations

Let M be a real analytic manifold. Let U be a locally closed subanalytic subset in M . Let \overline{U} be the closure of U in M . We have the bordered space $\mathbf{U} = (U, \overline{U})$. Let $\iota_U : U \rightarrow M$ denote the inclusion.

Let L be a local system on U . An object K of $E_{\mathbb{R}-c}^b(IC_U)$ with an isomorphism $\iota_K : E\iota_U^{-1}K \simeq \mathbb{C}_M^E \overset{+}{\otimes} \pi^{-1}(L)$ in $E^b(IC_U)$ is called a prolongation of L in $E_{\mathbb{R}-c}^b(IC_U)$. A morphism $\varphi : (K_1, \iota_{K_1}) \rightarrow (K_2, \iota_{K_2})$ of prolongations of L is a morphism $\varphi : K_1 \rightarrow K_2$ in $E_{\mathbb{R}-c}^b(IC_U)$ such that $\iota_{K_2} \circ \varphi|_U = \iota_{K_1}$.

We consider the following condition on prolongations $K \in E_{\mathbb{R}-c}^b(IC_M)$ of a local system on U .

- Set $I := [0, 1[$ and $I^\circ := I \setminus \{0\}$. Let $\varphi : I \rightarrow M$ be any real analytic map such that $\varphi(I^\circ) \subset U$. Then, we have subanalytic functions h_1, \dots, h_m on (I°, I) such that $E\varphi^{-1}K \simeq \bigoplus_{j=1}^m \mathbb{C}_\Delta^E \overset{+}{\otimes} \mathbb{C}_{t \geq h_j}$ in $E_{\mathbb{R}-c}^b(IC_{(I^\circ, I)})$.

Let $\text{Pro}_{tf}(U, M)$ denote the category of prolongations of local systems satisfying the above condition.

3.3.2 Filtrations by subanalytic subsets

We recall Lemma [3, Lemma 4.9.9]. Let M be a real analytic manifold. For any $K \in E_{\mathbb{R}-c}^b(IC_M)$, we have the following.

- A filtration $M = M^{(0)} \supset M^{(1)} \supset M^{(2)} \supset \dots$ by open subsets such that $M^{(i)} \setminus M^{(i+1)}$ are subanalytic submanifold of M with codimension i . Let $M^{(i)} \setminus M^{(i+1)} = \coprod_{j \in \Lambda(i)} \mathcal{C}_j^{(i)}$ be the decomposition into the connected components.
- For any $m \in \mathbb{Z}$, $i \in \mathbb{Z}_{\geq 0}$ and $j \in \Lambda(i)$, we have subanalytic functions $g_{j,k,m}^{(i)}$ ($k \in \Gamma_1(i, j, m)$) and $\psi_{j,\ell,m}^{(i)} < \phi_{j,\ell,m}^{(i)}$ ($\ell \in \Gamma_2(i, j, m)$) on $(\mathcal{C}_j^{(i)}, M)$ such that

$$\pi^{-1}(\mathbb{C}_{\mathcal{C}_j^{(i)}}) \otimes K \simeq \bigoplus_{m \in \mathbb{Z}} \bigoplus_{k \in \Gamma_1(i, j, m)} \mathbb{C}_M^E \overset{+}{\otimes} \mathbb{C}_{t \geq g_{j,k,m}^{(i)}}[m] \oplus \bigoplus_{m \in \mathbb{Z}} \bigoplus_{\ell \in \Gamma_2(i, j, m)} \mathbb{C}_M^E \overset{+}{\otimes} \mathbb{C}_{\psi_{j,\ell,m}^{(i)} \leq t < \phi_{j,\ell,m}^{(i)}}[m].$$

Such a filtration is called a filtration for K .

Lemma 3.19 *For any $K \in \text{Pro}_{tf}(U, M)$, we have the following.*

- We have a filtration $U = U^{(0)} \supset U^{(1)} \supset U^{(2)} \supset \dots$ such that $U^{(i)} \setminus U^{(i+1)}$ are locally closed subanalytic submanifolds of M with codimension i in M . We have the decomposition $U^{(i)} \setminus U^{(i+1)} = \coprod \mathcal{C}_j^{(i)}$ into connected components.
- We have subanalytic functions $g_{j,k}^{(i)}$ ($k \in \Gamma(i, j)$) on $(\mathcal{C}_j^{(i)}, M)$ such that the following holds in $\mathbb{E}_{\mathbb{R}\text{-c}}^b(IC_M)$:

$$\pi^{-1}(\mathbb{C}_{\mathcal{C}_j^{(i)}}) \otimes i_{U!!} K \simeq \bigoplus_{k \in \Gamma(i, j)} \mathbb{C}_M^{\mathbb{E}} \otimes^+ \mathbb{C}_{t \geq g_{j,k}^{(i)}}$$

Here, $i_U : U \longrightarrow M$ denotes the inclusion of the bordered spaces.

Proof We take a filtration $M = M^{(0)} \supset M^{(1)} \supset \dots$ for $i_{U!!} K$ as above. We may regard it as a filtration of U . Let $M^{(i)} \setminus M^{(i+1)} = \coprod_{j \in \Lambda(i)} \mathcal{C}_j^{(i)}$ be the decomposition into connected components. For any $m \in \mathbb{Z}$, $i \in \mathbb{Z}_{\geq 0}$ and $j \in \Lambda(i)$, we have subanalytic functions $g_{j,k,m}^{(i)}$ ($k \in \Gamma_1(i, j, m)$) and $\psi_{j,\ell,m}^{(i)} < \phi_{j,\ell,m}^{(i)}$ ($\ell \in \Gamma_2(i, j, m)$) with an isomorphism

$$\pi^{-1}(\mathbb{C}_{\mathcal{C}_j^{(i)}}) \otimes i_{U!!} K \simeq \bigoplus_{m \in \mathbb{Z}} \bigoplus_{k \in \Gamma_1(i, j, m)} \mathbb{C}_M^{\mathbb{E}} \otimes^+ \mathbb{C}_{t \geq g_{j,k,m}^{(i)}}[m] \oplus \bigoplus_{m \in \mathbb{Z}} \bigoplus_{\ell \in \Gamma_2(i, j, m)} \mathbb{C}_M^{\mathbb{E}} \otimes^+ \mathbb{C}_{\psi_{j,\ell,m}^{(i)} \leq t < \phi_{j,\ell,m}^{(i)}}[m].$$

Let $\varphi : I \longrightarrow M$ be a real analytic map such that $\varphi(I^\circ) \subset \mathcal{C}_j^{(i)}$ and that φ is an injection. Then, we have

$$\pi^{-1}(\mathbb{C}_{\varphi(I^\circ)}) \otimes i_{U!!} K \simeq \mathbb{E}\varphi!! \mathbb{E}\varphi^{-1} K \simeq \bigoplus_{i=1}^N \mathbb{C}_M^{\mathbb{E}} \otimes^+ \mathbb{C}_{t \geq h_i}$$

for some subanalytic functions h_i on $(\varphi(I^\circ), M)$. Hence, we obtain $\Gamma_1(i, j, m) = \emptyset$ unless $m = 0$, and $\Gamma_2(i, j, m) = \emptyset$ for any m . ■

Let $\mathcal{H}^0 \mathbb{E}_{\mathbb{R}\text{-c}}^b(IC_M)$ denote the heart of $\mathbb{E}_{\mathbb{R}\text{-c}}^b(IC_M)$ with respect to the t -structure in [3, §4.6]. (See also [3, Lemma 4.9.5].)

Corollary 3.20 $i_{U!!} \text{Pro}_{tf}(U, M) \subset \mathcal{H}^0 \mathbb{E}_{\mathbb{R}\text{-c}}^b(IC_M)$. ■

3.3.3 Description as quotient

Let $(K, \iota_K) \in \text{Pro}_{tf}(U, M)$ be a prolongment of a local system L on U . Let $i_U : U \longrightarrow M$ be the inclusion, and set $L_M := \iota_{U!} L \in \text{Mod}(\mathbb{C}_M)$.

Lemma 3.21 Suppose that U is a relatively compact in M . Then, we have a continuous subanalytic function $G \in \text{Sub}(U, M)$ and an epimorphism $K_G := \mathbb{C}_M^{\mathbb{E}} \otimes^+ (\pi^{-1} L_M \otimes \mathbb{C}_{t \geq G}) \longrightarrow K$ in $\text{Pro}_{tf}(U, M)$. Such a morphism $K_G \longrightarrow K$ is unique.

Proof We have the following:

- a filtration $U = U^{(0)} \supset U^{(1)} \supset U^{(2)} \supset \dots$ of U by open subanalytic subsets such that $U^{(i)} \setminus U^{(i+1)}$ are submanifolds of codimension i ,
- subanalytic functions $g_{j,k}^{(i)}$ ($k \in \Gamma(i, j)$) on connected components $\mathcal{C}_j^{(i)}$ of $U^{(i)} \setminus U^{(i+1)}$,
- isomorphisms $\pi^{-1}(\mathbb{C}_{\mathcal{C}_j^{(i)}}) \otimes K \simeq \bigoplus_{k \in \Gamma(i, j)} \mathbb{C}_M^{\mathbb{E}} \otimes^+ \mathbb{C}_{t \geq g_{j,k}^{(i)}}$.

For any relatively compact subset $V \subset U$, the restriction of $|g_{j,k}^{(i)}|$ to $V \cap \mathcal{C}_j^{(i)}$ are bounded.

By [3, Lemma 4.6.3, Proposition 4.7.2], we have an \mathbb{R} -constructible sheaf \mathfrak{F} on $\mathbb{R}_\infty \times U$ such that $\mathbb{C}_{t \geq 0}^+ \otimes \mathfrak{F} \simeq \mathfrak{F}$ and $K \simeq \mathbb{C}_M^E \otimes^+ \mathfrak{F}$. We may assume that the filtration $U^{(i)}$ and subanalytic functions $g_{j,k}^{(i)}$ are obtained from \mathfrak{F} , i.e., we may assume to have the following isomorphisms (see the proof of [3, Lemma 4.9.9]):

$$\pi^{-1}(\mathbb{C}_{\mathcal{C}_j^{(i)}}) \otimes \mathfrak{F} \simeq \bigoplus_{k \in \Gamma(i,j)} \mathbb{C}_{t \geq g_{j,k}^{(i)}}. \quad (14)$$

By using Lemma 2.9 and Lemma 2.10, we can take a continuous subanalytic function $G \in \text{Sub}(U, M)$ such that $G|_{\mathcal{C}_j^{(i)}} < g_{j,k}^{(i)}$ for any i, j, k . By (14), we have $\mathbb{C}_{t < G} \otimes \mathfrak{F} = 0$.

We have the isomorphism $\Phi : \mathbb{C}_U^E \otimes^+ (\pi^{-1}(L) \otimes \mathbb{C}_{t \geq 0}) \simeq (\mathbb{C}_M^E \otimes^+ \mathfrak{F})|_U$ in $\mathbb{E}_{\mathbb{R}-c}^b(IC_U)$. Let $V \subset U$ be any relatively compact open subset. Then, $\Phi|_V$ corresponds to a morphism

$$\Psi_{V,a(V)} : \pi^{-1}(\mathbb{C}_V) \otimes \pi^{-1}(L) \otimes \mathbb{C}_{t \geq -a(V)} \longrightarrow \pi^{-1}(\mathbb{C}_V) \otimes \mathfrak{F}$$

for a sufficiently large number $a(V) > 0$. Because $\mathbb{C}_{t < G} \otimes \mathfrak{F} = 0$, we have the following morphism induced by the morphism $\Psi_{V,a(V)}$:

$$\Psi_{V,G} : \pi^{-1}(\mathbb{C}_V) \otimes \pi^{-1}(L) \otimes \mathbb{C}_{t \geq G} \longrightarrow \pi^{-1}(\mathbb{C}_V) \otimes \mathfrak{F}$$

By enlarging V , we obtain the morphism $\Psi_G : \pi^{-1}(L) \otimes \mathbb{C}_{t \geq G} \longrightarrow \mathfrak{F}$ on $\mathbb{R}_\infty \times U$. It induces the following morphism in $\text{Pro}_{tf}(U, M)$:

$$K_G = \mathbb{C}_M^E \otimes^+ (\pi^{-1}(L_M) \otimes \mathbb{C}_{t \geq G}) \xrightarrow{\rho} \mathbb{C}_M^E \otimes^+ \mathfrak{F} = K.$$

We have $i_{U!!}K, i_{U!!}K_G \in \mathcal{H}^0 \mathbb{E}_{\mathbb{R}-c}^b(M)$. We set

$$\mathcal{F}^{(i)} i_{U!!}K := \pi^{-1}(\mathbb{C}_{U^{(i)}}) \otimes i_{U!!}K, \quad \mathcal{F}^{(i)} i_{U!!}K_G := \pi^{-1}(\mathbb{C}_{U^{(i)}}) \otimes i_{U!!}K_G.$$

By Lemma 3.6, they give filtrations in the abelian category $\mathcal{H}^0 \mathbb{E}_{\mathbb{R}-c}^b(M)$. By construction, the morphisms

$$\text{Gr}_{\mathcal{F}}^{(i)}(\rho) : \text{Gr}_{\mathcal{F}}^{(i)} i_{U!!}K_G \longrightarrow \text{Gr}_{\mathcal{F}}^{(i)} i_{U!!}K$$

are equal to the morphisms induced by the epimorphisms $\mathbb{C}_{t \geq G_j^{(i)}} \longrightarrow \mathbb{C}_{t \geq g_{j,k}^{(i)}}$, where $G_j^{(i)}$ denotes the restriction of G to $\mathcal{C}_j^{(i)}$. Because the functor $\mathfrak{G} \mapsto \mathbb{C}_M^E \otimes^+ \mathfrak{G}$ is exact and preserves the t -structure, we obtain that $\text{Gr}_{\mathcal{F}}^{(i)}(\rho)$ are epimorphisms. Hence, we obtain that ρ is an epimorphism.

Let $\rho' : K_G \longrightarrow K$ be any morphism in $\text{Pro}_{tf}(U, M)$. It is induced by a morphism $\pi^{-1}(L) \otimes \mathbb{C}_{t \geq G-a} \longrightarrow \mathfrak{F}$ in $\text{Mod}(\mathbb{C}_{\mathbb{R} \times U})$ for some $a > 0$. We have the induced morphism $\Psi'_G : \pi^{-1}(L) \otimes \mathbb{C}_{t \geq G} \longrightarrow \mathfrak{F}$, which also induces ρ' . Note that the induced morphisms $\mathbb{C}_M^E \otimes^+ \Psi'_G$ and $\mathbb{C}_M^E \otimes^+ \Psi_G$ are the same in $\mathbb{E}^b(IC_U)$. Hence, we obtain that $\Psi_G = \Psi'_G$. \blacksquare

3.3.4 Prolongations of isomorphisms

Proposition 3.22 *Let $(K_i, \iota_{K_i}) \in \text{Pro}_{tf}(U, M)$ ($i = 1, 2$). We have an isomorphism $(K_1, \iota_{K_1}) \simeq (K_2, \iota_{K_2})$ in $\text{Pro}_{tf}(U, M)$ if and only if the following holds.*

- Let $\varphi : I \longrightarrow M$ be any real analytic map such that $\varphi(I^\circ) \subset U$. Then, $\mathbb{E}\varphi^{-1}(K_1, \iota_1) \simeq \mathbb{E}\varphi^{-1}(K_2, \iota_2)$ in $\text{Pro}(I^\circ, I)$.

Such an isomorphism is unique if it exists.

Proof It is enough to consider the case where U is relatively compact. The “only if” part is clear. Let us prove the “if” part. Let L denote the local system on U such that K_a are prolongations of L . We have a continuous subanalytic function G and epimorphisms $K_G := \mathbb{C}_M^E \otimes^+ (\pi^{-1}(L) \otimes \mathbb{C}_{t \geq G}) \xrightarrow{\rho_a} K_a$ in $\text{Pro}_{tf}(U, M)$. It is enough to prove $\text{Ker}(\rho_1) = \text{Ker}(\rho_2)$.

We have a filtration $U = U^{(0)} \supset U^{(1)} \supset \dots$ for both K_1 and K_2 , by open subsets such that $U^{(i)} \setminus U^{(i+1)}$ are locally closed subanalytic submanifolds of M with codimension i . Let $U^{(i)} \setminus U^{(i+1)} = \coprod_{j \in \Lambda(i)} \mathcal{C}_j^{(i)}$ be the decomposition into connected components. For $a = 1, 2$, we have subanalytic functions $g_{a,j,p}^{(i)}$ ($p \in \Gamma(a, i, j)$) such that

$$\pi^{-1}(\mathbb{C}_{\mathcal{C}_j^{(i)}}) \otimes K_a \simeq \bigoplus_{p \in \Gamma(a, i, j)} \mathbb{C}_M^E \otimes^+ \mathbb{C}_{t \geq g_{a,j,p}^{(i)}}.$$

By Lemma 3.18, the isomorphisms $L|_{\mathcal{C}_j^{(i)}} \simeq L|_{\mathcal{C}_j^{(i)}}$ are extended to isomorphisms $\pi^{-1}(\mathbb{C}_{\mathcal{C}_j^{(i)}}) \otimes K_1 \simeq \pi^{-1}(\mathbb{C}_{\mathcal{C}_j^{(i)}}) \otimes K_2$.

We set $\mathcal{F}^{(i)}N := \pi^{-1}(\mathbb{C}_{U^{(i)}}) \otimes N$ for $N \in \mathcal{H}^0 \mathbb{E}_{\mathbb{R}-c}^b(M)$. They give filtrations \mathcal{F} on N in the category $\mathcal{H}^0 \mathbb{E}_{\mathbb{R}-c}^b(M)$. We have $\text{Ker Gr}_{\mathcal{F}}^{(i)}(\rho_1) = \text{Ker Gr}_{\mathcal{F}}^{(i)}(\rho_2)$ in $\text{Gr}_{\mathcal{F}}^{(i)} K_G$. By Lemma 3.5 and Lemma 3.6, we obtain $\text{Gr}_{\mathcal{F}}^{(i)} \text{Ker}(\rho_1) = \text{Gr}_{\mathcal{F}}^{(i)} \text{Ker}(\rho_2)$ in $\text{Gr}_{\mathcal{F}}^{(i)} K_G$. Hence, the claim of the “if” part follows from Lemma 3.23 below. The uniqueness of isomorphisms follows from the uniqueness in Lemma 3.21. \blacksquare

3.3.5 Appendix

Let U be a locally closed relatively compact subanalytic subset in M . Let \overline{U} be the closure in M . We set $U = (U, \overline{U})$. Let $Z \subset \overline{U}$ be a subanalytic closed subset. We set $U_1 := U \setminus Z$.

Let $\iota_U : U \rightarrow \overline{U}$ be the inclusion of the bordered spaces. Let N_0 be an object in $\mathcal{H}^0 \mathbb{E}_{\mathbb{R}-c}^b(\overline{U})$ obtained as $\iota_{U!} N'_0$ for an object $N'_0 \in \mathbb{E}_{\mathbb{R}-c}^b(U)$. Let N_a ($a = 1, 2$) be subobjects of N_0 in $\mathcal{H}^0 \mathbb{E}_{\mathbb{R}-c}^b(\overline{U})$ such that $\pi^{-1}(\mathbb{C}_U) \otimes N_a \simeq N_a$. Set $F^1 N_a := \pi^{-1}(\mathbb{C}_{U_1}) \otimes N_a$. We have the exact sequence

$$0 \rightarrow \pi^{-1}(\mathbb{C}_{U_1}) \otimes N_a \rightarrow N_a \rightarrow \pi^{-1}(\mathbb{C}_Z) \otimes N_a \rightarrow 0.$$

Suppose that $\pi^{-1}(\mathbb{C}_Z) \otimes N_1 = \pi^{-1}(\mathbb{C}_Z) \otimes N_2$ in $\pi^{-1}(\mathbb{C}_Z) \otimes N_0$, and $\pi^{-1}(\mathbb{C}_{U_1}) \otimes N_1 = \pi^{-1}(\mathbb{C}_{U_1}) \otimes N_2$ in $\pi^{-1}(\mathbb{C}_{U_1}) \otimes N_0$.

Lemma 3.23 *Under the assumption, we have $N_1 = N_2$.*

Proof It is enough to prove that the induced morphism $N_1 \rightarrow N_0/N_2$ is 0. It is enough to prove that the induced morphism $\pi^{-1}(\mathbb{C}_Z) \otimes N_1 \rightarrow \pi^{-1}(\mathbb{C}_{U_1}) \otimes (N_0/N_2)$ is 0.

We have $N_1 = \mathbb{C}_M^E \otimes^+ \mathfrak{F}$ and $N_0/N_2 = \mathbb{C}_M^E \otimes^+ \mathfrak{G}$. A morphism $\pi^{-1}(\mathbb{C}_Z) \otimes N_1 \rightarrow \pi^{-1}(\mathbb{C}_{U_1}) \otimes (N_0/N_2)$ corresponds to $\mathbb{C}_{t \geq -a} \otimes^+ (\pi^{-1}(\mathbb{C}_Z) \otimes \mathfrak{F}) \rightarrow \pi^{-1}(\mathbb{C}_{U_1}) \otimes \mathfrak{G}$ for some $a > 0$, which has to be 0 by the support condition. Hence, we obtain the desired vanishing. \blacksquare

3.4 A condition for the existence of global filtration

Let U_1 be an open ball in \mathbb{R}^{n-1} . We put $I := [0, 1[$ and $I^\circ := I \setminus \{0\}$. We put $U := I^\circ \times U_1$ and $\overline{U} := I \times U_1$. We use the coordinate (r, \mathbf{x}) of $\mathbb{R} \times \mathbb{R}^{n-1}$. For any ramified analytic function $f = \sum f_\eta(\mathbf{x}) r^\eta$, set $\text{ord}_r(f) := \min\{\eta \mid f_\eta \neq 0\}$.

Let $\mathcal{J} = \{f^{(1)}, \dots, f^{(m)}\}$ be a finite tuple of ramified analytic functions on U of the form $f^{(j)} = \sum f_\eta^{(j)} r^\eta$, where r is the coordinate of I , and $f_\eta^{(j)}$ are analytic functions on U_1 such that the following holds:

- If $j \neq k$, we have $a(j, k) := \text{ord}_r(f^{(j)} - f^{(k)}) < 0$, and $f_{a(j,k)}^{(j)} - f_{a(j,k)}^{(k)}$ is nowhere vanishing function on U_1 .

Let $\mathbf{m}(1), \dots, \mathbf{m}(m)$ be non-negative integers. Let $K \in \text{Pro}_{t_f}(U, \overline{U})$ be a prolongation of L such that the following holds:

- Let $\varphi : [0, 1[\rightarrow \overline{U}$ be any real analytic map such that $\varphi([0, 1[) \subset U$ and $\varphi(0) \in \overline{U} \setminus U$. Then, $\mathbb{E}\varphi^{-1}(K)$ is a stable free enhanced ind-sheaf induced by the local system $\varphi^{-1}(L)$ with a filtration \mathcal{F} indexed by $\varphi^* \mathcal{J}$ such that $\text{rank Gr}_{\varphi^{-1} \mathcal{F}(j)}^{\mathcal{F}} \varphi^{-1}(L) = \mathbf{m}(j)$.

Let us prove the following proposition.

Proposition 3.24 *We have a global filtration \mathcal{F} on L indexed by \mathcal{J} such that K is induced by L and a filtration \mathcal{F} .*

3.4.1 Preliminary

Let $\mathcal{C} \subset \mathbb{R}^n$ be a locally closed subanalytic subset. We consider tuples $\mathcal{J}_1 = \{f_1, \dots, f_m\}, \mathcal{J}_2 = \{g_1, \dots, g_\ell\} \subset \text{Sub}(\mathcal{C}, \mathbb{R}^n)$.

Lemma 3.25 *We assume the following.*

- Let $\gamma : I \rightarrow \mathbb{R}^n$ be any analytic map such that $\gamma(I^\circ) \subset \mathcal{C}$. Then, we have $\overline{\gamma^* \mathcal{J}_1} = \overline{\gamma^* \mathcal{J}_2}$ in $\overline{\text{Sub}}(I^\circ, I)$.

Then, we have a stratification $\mathcal{C} = \coprod \mathcal{C}_i$ such that $\overline{\mathcal{J}_1|_{\mathcal{C}_i}} = \overline{\mathcal{J}_2|_{\mathcal{C}_i}}$ in $\overline{\text{Sub}}(\mathcal{C}_i, \mathbb{R}^n)$ for any i .

Proof We have a rectilinearization $\phi_\alpha : W_\alpha \rightarrow \mathbb{R}^n$ ($\alpha \in \Lambda$) for \mathcal{C} and the functions $f_i, g_j, f_i - g_j, f_i - f_j, g_i - g_j$. Let Q be a quadrant $W_\alpha \cap \phi_\alpha^{-1}(\mathcal{C})$. Let us observe that $\overline{\{\phi_\alpha^*(f_i)|_Q\}} = \overline{\{\phi_\alpha^*(g_j)|_Q\}}$ holds. We may assume to have a decomposition $\{1, \dots, n\} = \{1, \dots, m_1\} \sqcup \{m_1 + 1, \dots, n\}$ such that $Q = \{x_i > 0 \ (i \leq m_1), \ x_i = 0 \ (i > m_1)\}$.

If $\phi_\alpha^*(f_i)|_Q - \phi_\alpha^*(f_j)|_Q$ is not constantly 0, we have

$$\phi_\alpha^*(f_i)|_Q - \phi_\alpha^*(f_j)|_Q = a_{i,j}^{(0)} \cdot \prod_{p \leq m_1} x_p^{\alpha(i,j)_p^{(0)}},$$

where $a_{i,j}^{(0)}$ is nowhere vanishing, and $(\alpha(i,j)_p^{(0)}) \in (\mathbb{Q}_{\geq 0})^{m_1} \cup (\mathbb{Q}_{\leq 0})^{m_1}$. Hence, $\overline{\{\phi_\alpha^*(f_j)|_Q\}} \subset \overline{\text{Sub}}(Q, \mathbb{R}^n)$ is totally ordered. By a similar argument, we obtain that $\overline{\{\phi_\alpha^*(f_i)|_Q\}} \cup \overline{\{\phi_\alpha^*(g_j)|_Q\}}$ is totally ordered.

Suppose that $\overline{\{\phi_\alpha^*(f_i)|_Q\}}$ is not contained in $\overline{\{\phi_\alpha^*(g_j)|_Q\}}$. We have

$$\phi_\alpha^*(f_i)|_Q - \phi_\alpha^*(g_j)|_Q = a_{i,j} \cdot \prod_{p \leq m_1} x_p^{\alpha(i,j)_p},$$

where $a_{i,j}$ is nowhere vanishing, and $(\alpha(i,j)_p) \in (\mathbb{Q}_{\leq 0})^{m_1} \setminus \{(0, \dots, 0)\}$.

Let $\phi_\alpha^*(g_{j_1})|_Q$ be a representative of the maximum of $\overline{\{\phi_\alpha^*(g_j)|_Q \mid \phi_\alpha^*(g_j)|_Q \prec \phi_\alpha^*(f_i)|_Q\}}$. Let $\phi_\alpha^*(g_{j_2})|_Q$ be a representative of the minimum of $\overline{\{\phi_\alpha^*(g_j)|_Q \mid \phi_\alpha^*(f_i)|_Q \prec \phi_\alpha^*(g_j)|_Q\}}$. Note that either one of the following holds: (1) $\alpha(i, j_1)_p \leq \alpha(i, j_2)_p$ for any p , or (2) $\alpha(i, j_1)_p \geq \alpha(i, j_2)_p$ for any p . Hence, we have p such that $\alpha(i, j_1)_p < 0$ and $\alpha(i, j_2)_p < 0$. Take $(c_1, \dots, c_{p-1}, c_{p+1}, \dots, c_{m_1}) \in \mathbb{R}_{>0}^{m_1-1}$. Let $\gamma_p : I_\epsilon \rightarrow Q$ be a real analytic map such that $\gamma_p(t) = (c_1, \dots, c_{p-1}, t, c_{p+1}, \dots, c_{m_1}, 0, \dots, 0)$. Then, we have $\gamma_p^* \phi_\alpha^*(g_j) \neq \gamma_p^* \phi_\alpha^*(f_i)$ in $\overline{\text{Sub}}(I^\circ, I)$ for any j , which contradicts the assumption. Hence, we obtain $\phi_\alpha^*(f_i)|_Q \in \overline{\{\phi_\alpha^*(g_j)|_Q\}}$. Similarly, we obtain $\phi_\alpha^*(g_j)|_Q \in \overline{\{\phi_\alpha^*(f_i)|_Q\}}$, and thus $\overline{\{\phi_\alpha^*(f_i)|_Q\}} = \overline{\{\phi_\alpha^*(g_j)|_Q\}}$.

We have subanalytic compact subsets $N_\alpha \subset W_\alpha$ such that $\bigcup_{\alpha \in \Lambda} \phi_\alpha(N_\alpha) = \mathbb{R}^n$. We can describe \mathcal{C} as the union of the subanalytic subsets $\mathcal{N}(\alpha, Q) := \phi_\alpha(Q \cap N_\alpha)$, where α runs through Λ , and Q runs through the set of quadrants in W_α such that $Q \subset \phi_\alpha^{-1}(\mathcal{C})$. We have $\overline{\mathcal{J}_1|_{\mathcal{N}(\alpha, Q)}} = \overline{\mathcal{J}_2|_{\mathcal{N}(\alpha, Q)}}$ in $\overline{\text{Sub}}(\mathcal{N}(\alpha, Q), \mathbb{R}^n)$. By using [14, Lemma 8.3.21], we can construct a stratification with the desired property which is finer than the covering $\mathcal{C} = \bigcup \mathcal{N}(\alpha, Q)$. \blacksquare

3.4.2 Proof of Proposition 3.24

By Lemma 3.25, we have a subanalytic stratification $U = \coprod \mathcal{C}$ with an isomorphism

$$\pi^{-1}(\mathbb{C}_{\mathcal{C}}) \otimes K \simeq \bigoplus \left(\mathbb{C}^E \otimes^+ \mathbb{C}_{t \geq f|_{\mathcal{C}}^{(j)}} \right) \otimes \mathbb{C}^{\mathbf{m}(j)}.$$

We have the corresponding filtration $\mathcal{F}^{\mathcal{C}}$ of $L|_{\mathcal{C}}$ indexed by \mathcal{J} such that $\text{rank Gr}_{f^{(j)}}^{\mathcal{F}}(L|_{\mathcal{C}}) = \mathbf{m}(j)$.

Lemma 3.26 *We have a filtration \mathcal{F} on L whose restriction to \mathcal{C} is equal to the filtration $\mathcal{F}^{\mathcal{C}}$.*

Proof Let \mathcal{C}_1 be an n -dimensional stratum. Let \mathcal{C}_2 be a stratum contained in the closure of \mathcal{C}_1 . By using a rectilinearization of the subanalytic sets \mathcal{C}_1 and \mathcal{C}_2 , we can find a real analytic map $\phi : [0, 1] \times [0, 1] \rightarrow \overline{U}$ such that the following holds:

- $\phi(a, b) \in \mathcal{C}_1$ if $a > 0, b > 0$.

- $\phi(a, 0) \in \mathcal{C}_2$ for any a .
- $\phi(0, b) \in \overline{U} \setminus U$ for any b .

Set $Z := [0, 1] \times [0, 1]$ and $Z^\circ := Z \setminus \{0\} \times [0, 1]$. We have the bordered space $\mathbf{Z} = (Z^\circ, Z)$. Let us study $\mathbf{E}\phi^{-1}K$ in $\mathbf{E}_{\mathbb{R}\text{-c}}^b(\mathbf{Z})$.

Set $\mathcal{U} := Z^\circ \setminus [0, 1] \times \{0\}$ and $Y^\circ := [0, 1] \times \{0\}$. We set $h^{(j)} := \phi^*(f^{(j)})$. The restrictions to \mathcal{U} and Y° are denoted by $h_{\mathcal{U}}^{(j)}$ and $h_{Y^\circ}^{(j)}$, respectively.

We have the filtration $\mathcal{F}^{\mathcal{U}}$ of $\phi^{-1}(L)|_{\mathcal{U}}$ indexed by $\{h_{\mathcal{U}}^{(j)}\}$ which is induced by $\mathcal{F}^{\mathcal{C}_1}$. It is naturally extended to a filtration \mathcal{F}^{Z° of $\phi^{-1}(L)$ indexed by $\{h^{(j)}\}$. By taking the restriction to Y° , we obtain the filtration $\mathcal{F}_{|Y^\circ}^{Z^\circ}$ of $\phi^{-1}(L)|_{Y^\circ}$. We also have the filtration \mathcal{F}^{Y° of $\phi^{-1}(L)|_{Y^\circ}$ indexed by $\{h_{Y^\circ}^{(j)}\}$ which is induced by $\mathcal{F}^{\mathcal{C}_2}$. It is enough to prove $\mathcal{F}_{|Y^\circ}^{\mathcal{U}} = \mathcal{F}^{Y^\circ}$.

We have an \mathbb{R} -constructible sheaf N on $(\mathbb{R}\overline{\mathbb{R}}) \times Z$ such that $\mathbb{C}^E \overset{+}{\otimes} N \simeq \mathbf{E}\phi^{-1}(K)$ and $\mathbb{C}_{t \geq 0} \overset{+}{\otimes} N \simeq N$. We have the following morphisms:

$$\mathbb{C}^E \overset{+}{\otimes} (\pi^{-1}(\mathbb{C}_{\mathcal{U}}) \otimes N) \simeq \bigoplus \mathbb{C}^E \overset{+}{\otimes} (\mathbb{C}_{t \geq h_{\mathcal{U}}^{(j)}} \otimes \mathbb{C}^{\mathbf{m}(j)}) \longrightarrow \bigoplus \mathbb{C}^E \overset{+}{\otimes} (\mathbb{C}_{t \geq h^{(j)}} \otimes \mathbb{C}^{\mathbf{m}(j)}).$$

We may assume that it is induced by a morphism $\pi^{-1}(\mathbb{C}_{\mathcal{U}}) \otimes N \longrightarrow \bigoplus \mathbb{C}_{t \geq h^{(j)}} \otimes \mathbb{C}^{\mathbf{m}(j)}$, which is extended to $N \longrightarrow \bigoplus \mathbb{C}_{t \geq h^{(j)}} \otimes \mathbb{C}^{\mathbf{m}(j)}$. Hence, we have a morphism

$$\mathbf{E}\phi^{-1}(K) \longrightarrow \bigoplus \mathbb{C}^E \overset{+}{\otimes} (\mathbb{C}_{t \geq h^{(j)}} \otimes \mathbb{C}^{\mathbf{m}(j)})$$

whose restriction to Z° is the identity of $\phi^{-1}(L)$. Set $Y := [0, 1] \times \{0\}$, and $\mathbf{Y} = (Y^\circ, Y)$. Let $\iota : \mathbf{Y} \longrightarrow \mathbf{Z}$ be the inclusion of bordered spaces. We obtain a morphism

$$\mathbf{E}(\phi \circ \iota)^{-1}K \longrightarrow \bigoplus \mathbb{C}^E \overset{+}{\otimes} (\mathbb{C}_{t \geq h_{Y^\circ}^{(j)}} \otimes \mathbb{C}^{\mathbf{m}(j)}).$$

whose restriction to Y° is the identity of $\phi^{-1}(L)|_{Y^\circ}$. It means we have $\mathcal{F}_{f^{(j)}}^{Y^\circ} \subset (\mathcal{F}_{f^{(j)}}^{\mathcal{U}})_{|Y^\circ}$ for any $f^{(j)}$. By comparing the rank, we obtain $\mathcal{F}_{f^{(j)}}^{Y^\circ} = (\mathcal{F}_{f^{(j)}}^{\mathcal{U}})_{|Y^\circ}$. ■

Then, the claim of Proposition 3.24 follows from Proposition 3.22. ■

3.5 Subcategories of \mathbb{R} -constructible enhanced ind-sheaves

Let X be a complex manifold with a hypersurface H . We have the bordered space $\mathbf{X}(H) := (X \setminus H, X)$. Let $\mathbf{E}_{\mathbb{R}\text{-c}}^b(IC_{\mathbf{X}(H)})$ be the category of \mathbb{R} -constructible enhanced ind-sheaves on $\mathbf{X}(H)$ [3, 4]. We introduce some subcategories of $\mathbf{E}_{\mathbb{R}\text{-c}}^b(IC_{\mathbf{X}(H)})$.

3.5.1 Subcategory of meromorphic flat connections

Let $\text{Hol}(\mathcal{D}_X)$ denote the category of holonomic \mathcal{D}_X -modules. Let $\text{Mero}_*(X, H) \subset \text{Hol}(\mathcal{D}_X)$ be the full subcategory of holonomic \mathcal{D}_X -modules M which are coherent over $\mathcal{O}_X(*H)$. According to [3], we have the de Rham functor $\text{DR}_{\mathbf{X}(H)}^E : \text{Mero}(X, H) \longrightarrow \mathbf{E}_{\mathbb{R}\text{-c}}^b(IC_{\mathbf{X}(H)})$, and it is a fully faithful functor. Let $\mathbf{E}_{\text{mero}}^b(IC_{\mathbf{X}(H)}) \subset \mathbf{E}_{\mathbb{R}\text{-c}}^b(IC_{\mathbf{X}(H)})$ denote the essential image of $V \longmapsto \text{DR}_{\mathbf{X}(H)}^E(V)[-d_X]$.

Let $\text{Mero}_!(X, H) \subset \text{Hol}(\mathcal{D}_X)$ denote the essential image of $\mathbf{D} : \text{Mero}_*(X, H)^{\text{op}} \longrightarrow \text{Hol}(\mathcal{D}_X)$, where \mathbf{D} denotes the duality functor on $\text{Hol}(\mathcal{D}_X)$, and $\text{Mero}_*(X, H)^{\text{op}}$ denotes the opposite category of $\text{Mero}_*(X, H)$. We may also regard $\mathbf{E}_{\text{mero}}^b(IC_{\mathbf{X}(H)})$ as the essential image of $\text{Mero}_!(X, H) \longrightarrow \mathbf{E}_{\mathbb{R}\text{-c}}^b(IC_{\mathbf{X}(H)})$ given by $M \longmapsto \text{DR}_{\mathbf{X}(H)}^E(M)[-d_X]$.

Let $j : (X \setminus H, X) \longrightarrow (X, X)$ be the natural inclusion of the bordered spaces. We have the fully faithful functor $Rj_! : \mathbf{E}_{\mathbb{R}\text{-c}}^b(IC_{\mathbf{X}(H)}) \longrightarrow \mathbf{E}_{\mathbb{R}\text{-c}}^b(IC_X)$. The essential image of the induced functor $Rj_! : \mathbf{E}_{\text{mero}}^b(IC_{\mathbf{X}(H)}) \longrightarrow \mathbf{E}_{\mathbb{R}\text{-c}}^b(IC_X)$ is equal to the essential image of $\text{Mero}_!(X, H) \longrightarrow \mathbf{E}_{\mathbb{R}\text{-c}}^b(IC_X)$ given by $M \longmapsto \text{DR}_X^E(M)[-d_X]$.

3.5.2 Subcategory determined by curve test

Let $\mathbf{E}_{\odot}^b(IC_{\mathbf{X}(H)}) \subset \mathbf{E}_{\mathbb{R}\text{-}c}^b(IC_{\mathbf{X}(H)})$ be the full subcategory of objects K satisfying the following conditions.

- $K|_{X \setminus D}$ is a locally free $\mathbb{C}|_{X \setminus D}$ -module.
- Let $\Delta(0)$ denote the bordered space $(\Delta \setminus \{0\}, \Delta)$. Let $\varphi : \Delta \rightarrow X$ be a holomorphic map such that $\varphi(0) \in D$ and $\varphi(\Delta \setminus \{0\}) \subset X \setminus D$. Then, $\mathbf{E}_{\varphi}^{-1}K$ is an object in $\mathbf{E}_{\text{mero}}^b(IC_{\Delta(0)})$.

By the compatibility of the de Rham functor and the 6-functors for holonomic \mathcal{D} -modules and \mathbb{R} -constructible enhanced ind-sheaves in [3], $\mathbf{E}_{\text{mero}}^b(IC_{\mathbf{X}(H)})$ is a full subcategory of $\mathbf{E}_{\odot}^b(IC_{\mathbf{X}(H)})$. If $\dim X = 1$, we clearly have $\mathbf{E}_{\text{mero}}^b(IC_{\mathbf{X}(H)}) = \mathbf{E}_{\odot}^b(IC_{\mathbf{X}(H)})$. We shall later prove $\mathbf{E}_{\text{mero}}^b(IC_{\mathbf{X}(H)}) = \mathbf{E}_{\odot}^b(IC_{\mathbf{X}(H)})$ even in the higher dimensional case.

3.5.3 Subcategory determined by sector test

For any $\epsilon > 0$, we set $\Delta_{\epsilon} := \{z \in \mathbb{C} \mid |z| < \epsilon\}$. Let $\varpi_0 : \tilde{\Delta}_{\epsilon}(0) \rightarrow \Delta_{\epsilon}$ denote the oriented real blowing up at 0, i.e., $\tilde{\Delta}_{\epsilon}(0) = [0, \epsilon[\times S^1$. Let $S_{\epsilon, \delta} \subset \tilde{\Delta}_{\epsilon}(0)$ denote the sector $\{(r, \theta) \in \tilde{\Delta}_{\epsilon}(0) \mid -\delta < \theta < \delta\}$. Set $Z_{\delta} := \{(0, \theta) \mid -\delta < \theta < \delta\} \subset S_{\epsilon, \delta}$, and $S_{\epsilon, \delta}^{\circ} := S_{\epsilon, \delta} \setminus Z_{\delta}$. We have $S_{\epsilon, \delta}^{\circ} = \{z \mid 0 < |z| < \epsilon, -\delta < \arg(z) < \delta\} \subset \mathbb{C}$. Let $S_{\epsilon, \delta}^{\circ}$ denote the bordered space $(S_{\epsilon, \delta}^{\circ}, S_{\epsilon, \delta})$.

We set $I := \{0 \leq t < 1\}$ and $I^* := I \setminus \{0\}$. Let X be a complex manifold. For any analytic map $\varphi : (I, 0) \rightarrow (X, H)$ such that $\varphi(I^*) \subset X \setminus H$, if ϵ is sufficiently small, we have the induced holomorphic map $\varphi_{\mathbb{C}} : \Delta_{\epsilon} \rightarrow X$. It induces the analytic map of bordered spaces $\tilde{\varphi}_{\epsilon, \delta} : S_{\epsilon, \delta}^{\circ} \rightarrow \mathbf{X}(H)$.

Let $\mathbf{E}_{\odot}^b(IC_{\mathbf{X}(H)}) \subset \mathbf{E}_{\mathbb{R}\text{-}c}^b(IC_{\mathbf{X}(H)})$ be the full subcategory of objects K satisfying the following conditions:

- Let $\varphi : (I, 0) \rightarrow (X, H)$ be any real analytic map such that $\varphi(I^*) \subset X \setminus H$. Then, we have small positive numbers δ and ϵ , and \mathbb{C} -valued subanalytic functions g_1, \dots, g_m on $(S_{\epsilon, \delta}^{\circ}, \tilde{\Delta}(0))$ which are holomorphic as functions on $S_{\epsilon, \delta}^{\circ}$, such that

$$\mathbf{E}_{\tilde{\varphi}_{\epsilon, \delta}}^{-1}K \simeq \bigoplus_{i=1}^m \mathbb{C}_{S_{\epsilon, \delta}^{\circ}}^{\mathbf{E}} \otimes \mathbb{C}_{t \geq \text{Re}(g_i)}.$$

The following is clear by the definitions.

Lemma 3.27 *We have $\mathbf{E}_{\odot}^b(\mathbf{X}_H) \subset \text{Pro}_{tf}(\mathbf{X}_H)$.* ■

3.6 Comparison of curve test and sector test

3.6.1 Basic isomorphisms

Let X be a complex manifold with a complex hypersurface H . Let f be a meromorphic function on (X, H) . We have the object $\mathcal{E}_X^f(*H) := (\mathcal{O}_X(*H), d + df)$ in $\text{Mero}_*(X, H)$. We also have $\mathcal{E}_X^f(!H) := \mathbf{D}_X(\mathcal{E}_X^{-f}(*H))$, where \mathbf{D}_X denotes the duality functor on $\text{Hol}(X)$. According to [3], we have natural isomorphisms in $\mathbf{E}_{\mathbb{R}\text{-}c}^b(IC_X)$:

$$\text{DR}_X^{\mathbf{E}}(\mathcal{E}_X^f(*H))[-d_X] \simeq \mathbb{C}_X^{\mathbf{E}} \otimes^+ R\mathcal{I}hom(\pi^{-1}(\mathbb{C}_{X \setminus H}), \mathbb{C}_{t=\text{Re}(f)}), \quad (15)$$

$$\text{DR}_X^{\mathbf{E}}(\mathcal{E}_X^f(!H))[-d_X] \simeq \mathbb{C}_X^{\mathbf{E}} \otimes^+ \mathbb{C}_{t \geq \text{Re}(f)}. \quad (16)$$

3.6.2 Comparison of curve test and sector test

Let X be a complex manifold with a hypersurface H .

Proposition 3.28 *The subcategories $\mathbf{E}_{\odot}(IC_{\mathbf{X}(H)})$ and $\mathbf{E}_{\odot'}(IC_{\mathbf{X}(H)})$ are the same. In particular, if $\dim X = 1$, we have $\mathbf{E}_{\text{mero}}(IC_{\mathbf{X}(H)}) = \mathbf{E}_{\odot'}(IC_{\mathbf{X}(H)})$.*

Proof We begin with a lemma for functions on one dimensional sectors.

Lemma 3.29 *Let $\mathcal{S} := \{z \in \mathbb{C} \setminus \{0\} \mid 0 < |z| < \epsilon, |\arg(z)| < \delta\}$. Let f be a continuous subanalytic \mathbb{C} -valued function on $(\mathcal{S}, \mathbb{C})$. Suppose that f is holomorphic on \mathcal{U} . Then, there exists a positive integer ρ and a germ of meromorphic function g at 0 such that $f|_{U \cap \mathcal{S}} = g(z^{1/\rho})|_{U \cap \mathcal{S}}$ for a sufficiently small neighbourhood U of 0.*

Proof Let $\varpi : \tilde{\mathbb{C}}(0) \rightarrow \mathbb{C}$ be the real blowing up. We may regard \mathcal{S} as an open subset in $\tilde{\mathbb{C}}(0)$. We may regard f as a subanalytic function on $(\mathcal{S}, \tilde{\mathbb{C}}(0))$. Let $\overline{\mathcal{S}}$ denote the closure of \mathcal{S} in $\tilde{\mathbb{C}}(0)$. We set $Z = \overline{\mathcal{S}} \cap \varpi^{-1}(0)$. Let Z° be the interior part of $Z \subset \varpi^{-1}(0)$. We can take a sufficiently large integer N such that $z^N f$ is extended to a continuous function on $\mathcal{S} \cup Z^\circ$, and $(z^N f)|_{Z^\circ} = 0$. We have a discrete subset $Z' \subset Z^\circ$ such that for any $\theta_0 \in Z \setminus Z'$ we have a neighbourhood \mathcal{U}_0 and a positive integer $\rho_0 > 0$ ($z^N f$) $|_{\mathcal{U}_0}$ is expressed by a convergent power series:

$$(z^N f)|_{\mathcal{U}_0} = \sum_{j>0} \sum_{i \geq 0} a_{i,j} (\theta - \theta_0)^i \cdot r^{j/\rho_0}.$$

Because $z^N f$ is holomorphic, we have $(z^N f)|_{\mathcal{U}_0} = \sum_{j>0} b_j z^{j/\rho_0}$. Then, the claim of the lemma follows. \blacksquare

To prove Proposition 3.28, it is enough to consider the case $X = \Delta$ and $H = \{0\}$. It follows from the following lemma.

Lemma 3.30 *An object K of $\mathbf{E}_{\mathbb{R}\text{-}c}^b(IC_{\mathbf{X}(H)})$ is an object in $\mathbf{E}_{\text{mero}}^b(IC_{\mathbf{X}(H)})$ if and only if we have the following:*

- a finite covering $X \setminus H = \bigcup_{i=1}^N \mathcal{S}_i$, where $\mathcal{S}_i = \{z \in \mathbb{C} \setminus \{0\} \mid 0 < |z| < \epsilon, \theta_{i,1} < \arg(z) < \theta_{i,2}\}$,
- subanalytic \mathbb{C} -valued functions g_{ij} ($j = 1, \dots, m$) on $(\mathcal{S}_i, \mathbb{C})$ such that g_{ij} are holomorphic on \mathcal{S}_i ,
- the following isomorphism in $\mathbf{E}_{\mathbb{R}\text{-}c}^b(IC_{\mathbf{X}(H)})$:

$$\pi^{-1}(\mathbb{C}_{\mathcal{S}_i}) \otimes K \simeq \bigoplus_{j=1}^m \pi^{-1}(\mathbb{C}_{\mathcal{S}_i}) \otimes \left(\mathbb{C}_X^E \overset{+}{\otimes} \mathbb{C}_{t \geq \text{Re}(g_{ij})} \right)$$

Proof The only if part follows from the basic isomorphisms (15, 16) and the classical asymptotic analysis for meromorphic flat bundles on curves. Let us study the if part. By shrinking \mathcal{S}_i , we may assume that g_{ij} are bounded on $\{\epsilon' < |z| < \epsilon, \theta_{i,1} < \theta < \theta_{i,2}\}$ for any $\epsilon' > 0$. By taking an appropriate ramified covering $\psi : (\Delta, 0) \rightarrow (\Delta, 0)$, we may assume that $\psi^* g_{ij}$ are extended to meromorphic functions on $(\Delta, 0)$, by Lemma 3.29. We may assume that they are elements in $z^{-1}\mathbb{C}[z^{-1}]$.

By the condition, $K|_{X \setminus D}$ gives a $\mathbb{C}_{X \setminus D}$ -locally free sheaf on $X \setminus D$, which is denoted by L . Let $\varpi : \tilde{X}(D) \rightarrow X$ denote the oriented real blowing up. We have the local system \tilde{L} on $\tilde{X}(D)$ induced by L .

We set $I_i := \{\text{Re}(g_{ij}) \mid j = 1, \dots, m\}$. Let $\overline{\mathcal{S}}_i$ denote the closure of \mathcal{S}_i in $\tilde{X}(D)$. Let \overline{I}_i denote the image of $I_i \rightarrow \overline{\text{Sub}}(\mathcal{S}_i, \overline{\mathcal{S}}_i)$.

As remarked in §3.2.4, we have the filtration $\mathcal{F}^{\mathcal{S}_i}$ on $L|_{\mathcal{S}_i}$ indexed by $(\overline{I}_i, <)$. If $\mathcal{S}_{i,j} := \mathcal{S}_i \cap \mathcal{S}_j \neq \emptyset$, we have $\overline{I}_i|_{\mathcal{S}_{i,j}} = \overline{I}_j|_{\mathcal{S}_{i,j}}$ in $\overline{\text{Sub}}(\mathcal{S}_{i,j}, \overline{\mathcal{S}}_{i,j})$. We also have the filtration $\mathcal{F}^{\mathcal{S}_i \cap \mathcal{S}_j}$ on $L|_{\mathcal{S}_i \cap \mathcal{S}_j}$ which is equal to the filtrations induced by $\mathcal{F}^{\mathcal{S}_k}$ ($k = i, j$). Hence, \tilde{L} is equipped with a Stokes structure \mathcal{F} , i.e., a family of Stokes filtrations.

Let (V, ∇) be the meromorphic flat bundle corresponding to (\tilde{L}, \mathcal{F}) . The isomorphism $\text{DR}_{\mathcal{S}_i}^E(V, \nabla)[-1] \simeq K|_{\mathcal{S}_i}$ is extended to isomorphisms $\pi^{-1}(\mathbb{C}_{\mathcal{S}_i}) \otimes \text{DR}_{\mathbf{X}(H)}^E(V, \nabla) \simeq \pi^{-1}(\mathbb{C}_{\mathcal{S}_i}) \otimes K$ in $\mathbf{E}_{\mathbb{R}\text{-}c}^b(IC_{\mathbf{X}(H)})$. Such isomorphisms are unique by Lemma 3.16. Hence, we have $\text{DR}_{\mathbf{X}(H)}^E(V, \nabla) \simeq K$. Thus, we obtain Lemma 3.30, and Proposition 3.28. \blacksquare

3.6.3 Prolongations of isomorphisms of enhanced ind-sheaves

Let X be a complex manifold with a hypersurface H . Let $(V, \nabla) \in \text{Mero}_*(X, H)$. Let L_V be the local system on $X \setminus H$ corresponding to $(V, \nabla)|_{X \setminus H}$. Let $K \in \mathbf{E}_{\odot}^b(IC_{\mathbf{X}(H)})$ such that $K|_{X \setminus H}$ comes from a locally free $\mathbb{C}_{X \setminus H}$ -module L_K . Suppose the following.

- We are given an isomorphism of $\mathbb{C}_{X \setminus H}$ -modules $\Phi : L_V \simeq L_K$.

- Let $\varphi : \Delta \rightarrow X$ be any holomorphic map such that $\varphi(\{0\}) \in H$ and $\varphi(\Delta \setminus \{0\}) \subset X \setminus H$. Then, $\varphi^{-1}\Phi$ is extended to an isomorphism $\mathrm{DR}_{\Delta(0)}^E(\varphi^*(V))[-1] \simeq E\varphi^{-1}K$ in $E_{\mathbb{R}-c}^b(IC_{\Delta(0)})$.

We obtain the following proposition as a consequence of Proposition 3.22 and Proposition 3.28.

Proposition 3.31 *Φ is extended to an isomorphism $\mathrm{DR}_{X(H)}^E(V)[-d_X] \simeq K$ in $E_{\mathbb{R}-c}^b(IC_{X(H)})$.* ■

3.7 Auxiliary conditions

3.7.1 Set of ramified irregular values

Let X be a complex manifold with a normal crossing hypersurface H . Let P be any point of H . Let (X_P, z_1, \dots, z_n) denote a small coordinate neighbourhood of P in X such that $H_P := H \cap X_P = \bigcup_{i=1}^{\ell} \{z_i = 0\}$. For any positive integer e , let $X_P^{(e)}$ be an open subset in $\mathbb{C}^n = \{(\zeta_1, \dots, \zeta_n)\}$, and let $\psi_e : X_P^{(e)} \rightarrow X_P$ be a ramified covering given by $\psi_e(\zeta_1, \dots, \zeta_n) = (\zeta_1^e, \dots, \zeta_{\ell}^e, \zeta_{\ell+1}, \dots, \zeta_n)$. Set $H_P^{(e)} = \psi_e^{-1}(H_P)$. A set of ramified irregular values at P is a tuple of meromorphic functions $\{g_1, \dots, g_m\}$ on $(X_P^{(e)}, H_P^{(e)})$ for some e , which is invariant by the action of the Galois group of ψ_e . If $\{g_1, \dots, g_m\}$ is a good set of irregular values on $(X_P^{(e)}, H_P^{(e)})$, then it is called a good set of ramified irregular values at P . A multiplicity function on $\{g_1, \dots, g_m\}$ be a function $\mathbf{m} : \{g_1, \dots, g_m\} \rightarrow \mathbb{Z}_{\geq 0}$.

3.7.2 Some conditions

We introduce a condition for objects $K \in E_{\odot}^b(IC_{X(H)})$. Let P be any point of H . If the following condition is satisfied, we say that K satisfies **(A)** at P :

- We are given a set of ramified irregular values \mathcal{I}_P at P , a multiplicity function $\mathbf{m}_P : \mathcal{I}_P \rightarrow \mathbb{Z}_{\geq 0}$, and a neighbourhood H_P of P in H . Note that \mathcal{I}_P is not necessarily a good set of ramified irregular values.
- For any holomorphic map $\varphi : \Delta \rightarrow X$ satisfying $\varphi(0) \in H_P$ and $\varphi(\Delta \setminus \{0\}) \subset X \setminus H$, we have $\mathrm{Irr}(E\varphi^{-1}(K)) = \varphi^*\mathcal{I}_P$ compatible with the multiplicity.

We say that K satisfies **(GA)** at P if \mathcal{I}_P is a good set of ramified irregular values.

We say that K satisfies **(A)** (resp. **(GA)**) if it satisfies **(A)** (resp. **(GA)**) at any point of H .

3.7.3 Good smooth case

Let X be an n -dimensional complex manifold. Let H be a smooth hypersurface of X . We consider $K \in E_{\odot}^b(IC_{X(H)})$ satisfying the condition **(GA)**.

Proposition 3.32 *We have a good meromorphic flat bundle (V, ∇) on (X, H) with an isomorphism*

$$\mathrm{DR}_{X(H)}^E(V)[-n] \simeq K.$$

Proof Let P be any point of H . We have a good set of ramified irregular values \mathcal{I}_P with a multiplicity function $\mathbf{m}_P : \mathcal{I}_P \rightarrow \mathbb{Z}_{\geq 0}$ as in the definition of the condition **(GA)**. We may assume that \mathcal{I}_P is unramified, i.e., elements of \mathcal{I}_P are meromorphic functions on (X_P, H_P) , where X_P is a small neighbourhood of P in X , and $H_P := X_P \cap H$. Set $K_P := K|_{X_P}$. Let L_P be the local system on $X_P \setminus H_P$ corresponding to $K|_{X_P \setminus H_P}$. We may assume $X_P = \Delta^n = \{(z_1, \dots, z_n) \mid |z_i| < 1\}$, $H_P = \{z_1 = 0\}$ and $P = (0, \dots, 0)$.

Let $\varphi_P : \Delta \rightarrow X_P$ be given by $\varphi_P(\zeta) = (\zeta, 0, \dots, 0)$. We have a meromorphic flat bundle $V_{P,0}$ with an isomorphism $\mathrm{DR}_{\Delta}^E V_{P,0}[-1] \simeq E\varphi_P^{-1}(K)$. We have $\mathrm{Irr}(V_{P,0}) = \varphi_P^*\mathcal{I}_P$ compatible with the multiplicity. By using [30], we obtain an unramifiedly good meromorphic flat bundle V_P on (X_P, H_P) such that $\mathrm{Irr}(V_P) = \mathcal{I}_P$ with an isomorphism $\varphi_P^* V_P \simeq V_{P,0}$. Such V_P is uniquely determined up to canonical isomorphisms.

Let $\varpi_P : \tilde{X}_P(H_P) \rightarrow X_P$ be the oriented real blowing up. For $f, g \in \mathcal{I}_P$ with $f \neq g$, we have the C^∞ -function $F_{f,g} := \mathrm{Re}(f - g)|z_1|^{-\mathrm{ord}(f-g)}$ on $\tilde{X}_P(H_P)$. Set Z as the union of $F_{f,g}^{-1}(0)$ for such f, g . We have $\dim_{\mathbb{R}} Z = 2n - 2$.

The object $\mathbb{E}\varpi_P^{-1}(K)$ on the bordered space $\widetilde{X}_P(H_P) = (X_P \setminus H_P, \widetilde{X}_P(H_P))$ is a prolongation of L_P . Take any $Q \in \varpi_P^{-1}(H_P) \setminus Z$. We take a small neighbourhood \mathcal{U}_Q of Q in $\widetilde{X}_P(H_P)$ such that $\mathcal{U}_Q \cap Z = \emptyset$. We set $\mathcal{U}_Q^\circ := \mathcal{U}_Q \setminus \varpi_P^{-1}(H_P)$. By Proposition 3.24, we have a filtration \mathcal{F}^Q of $L_P|_{\mathcal{U}_Q^\circ}$ such that $\mathbb{E}\varpi_P^{-1}(K_P)|_{\mathcal{U}_Q}$ is the stably free ind-sheaf induced by the local system $L_P|_{\mathcal{U}_Q^\circ}$ with the filtration \mathcal{F}^Q . By considering the restriction to $\varpi_P^{-1}(\Delta \times \{(0, \dots, 0)\})$, we obtain that there exists an isomorphism

$$\mathbb{E}\varpi_P^{-1} \mathrm{DR}_{X_P(H_P)}^{\mathbb{E}}(V_P)[-n]|_{\mathcal{U}_Q} \simeq \mathbb{E}\varpi_P^{-1}(K)|_{\mathcal{U}_Q}$$

extending the isomorphism of the local systems on \mathcal{U}_Q° .

Let $\varpi_P^{-1}(P) \setminus Z = \coprod \mathcal{I}_j$ be the decomposition into the connected components. We take a point Q_j for each \mathcal{I}_j . We take a neighbourhood \mathcal{U}_{Q_j} as above. By shrinking X_P , we assume $\varpi_P(\mathcal{U}_{Q_j}) = H_P$ for each j . We take any $P' = (0, z_2^{(0)}, \dots, z_n^{(0)}) \in H_P$, and let $\varphi_{P'} : \Delta \rightarrow X_P$ be the holomorphic map given by $\varphi_{P'}(\zeta) = (\zeta, z_2^{(0)}, \dots, z_n^{(0)})$. Then, by the comparison of the Stokes filtrations and by using Lemma 2.23, we obtain an isomorphism $\mathrm{DR}_{\Delta}^{\mathbb{E}} \varphi_{P'}^*(V_P)[-1] \simeq \mathbb{E}\varphi_{P'}^{-1}(K_P)$ extending the isomorphism of the local systems on $\Delta \setminus \{0\}$.

Let $\varphi : \Delta \rightarrow X_P$ be any holomorphic map such that $\varphi(\Delta \setminus \{0\}) \subset X_P \setminus H_P$ and $\varphi(0) \in H_P$. Similarly, by comparing the Stokes filtrations, we obtain an isomorphism $\mathrm{DR}_{\Delta}^{\mathbb{E}} \varphi^*(V_P)[-1] \simeq \mathbb{E}\varphi^{-1}(K_P)$ extending the isomorphism of the local systems on $\Delta \setminus \{0\}$.

By varying $P \in H$, and by gluing V_P , we obtain a good meromorphic flat bundle V on (X, H) . By using Proposition 3.31, we have an isomorphism $\mathrm{DR}_{X(H)}^{\mathbb{E}}(K)[-n] \simeq K$ extending the isomorphism of the local systems on $X \setminus H$. \blacksquare

4 Generic part of normal crossing hypersurfaces

4.1 Ramified irregular values outside of subset with real codimension one

Let $n > 1$ be an integer. Set $\Delta = \{z \in \mathbb{C} \mid |z| < 1\}$. Let H be an $(n-1)$ -dimensional complex manifold. We set $X := \Delta \times H$. We identify H and $\{0\} \times H$. Let $\varpi : \widetilde{X}(H) \rightarrow X$ be the oriented real blowing up. Let $K \in \mathbb{E}_{\odot}^b(IC_{X(H)})$. Take $P_0 \in H$ and $Q_0 \in \varpi^{-1}(P_0)$. We assume the following.

- We have a neighbourhood \mathcal{U}_{Q_0} of Q_0 in $\widetilde{X}(H)$ and analytic functions $h_1^{Q_0}, \dots, h_m^{Q_0}$ on $\mathcal{U}_{Q_0}^\circ = \mathcal{U}_{Q_0} \setminus \varpi^{-1}(H)$ such that (i) they are ramifiedly analytic around P (see §2.2.2 for ramified analyticity), (ii) they control the growth order of $\pi^{-1}(\mathbb{C}_{\mathcal{U}_{Q_0}^\circ}) \otimes K$, i.e., $\pi^{-1}(\mathbb{C}_{\mathcal{U}_{Q_0}^\circ}) \otimes K = \bigoplus_{i=1}^m \mathbb{C}_{\widetilde{X}(H)}^{\mathbb{E}} \otimes \mathbb{C}_{t \geq h_i^{Q_0}}$.

Lemma 4.1 *K satisfies (A) at P_0 .*

Proof We may assume that H is equipped with a holomorphic coordinate $\mathbf{w} = (w_1, \dots, w_{n-1})$. For any $\mathbf{w} \in H$, let $\varphi_{\mathbf{w}} : \Delta \rightarrow X$ be the holomorphic map given by $\varphi_{\mathbf{w}}(z) = (z, \mathbf{w})$. Because $\mathbb{E}\varphi_{\mathbf{w}}^{-1}(K) \in \mathbb{E}_{\mathrm{mero}}^b(IC_{\Delta(0)})$, we have the set of ramified irregular values $\mathrm{Irr}(\mathbb{E}\varphi_{\mathbf{w}}^{-1}K)$. We set $H_{P_0} := \varpi(\mathcal{U}_{Q_0})$. We may assume that \mathcal{U}_{Q_0} is the product of H_{P_0} and a closed sector in $\widetilde{\Delta}(0)$.

We have the holomorphic coordinate (z, w_1, \dots, w_{n-1}) of $X = \Delta \times H$. Let r and θ be given by the polar decomposition $z = re^{\sqrt{-1}\theta}$. By taking a ramified covering of (X, H) , we may assume that $h_i^{Q_0}$ are expressed as follows:

$$h_i^{Q_0} = \sum_{-N_1 \leq j < 0} \alpha_{i,j}(\theta, \mathbf{w}) \cdot r^j$$

Here, $\alpha_{i,j}$ are real analytic functions on a neighbourhood of Q_0 in $\partial\widetilde{X}(H) = S^1 \times H$. For each $\mathbf{w} \in H_{P_0}$, because

$$\varphi_{\mathbf{w}}^*(h_i^{Q_0}) = \sum_{-N_1 \leq j < 0} \alpha_{i,j}(\theta, \mathbf{w}) \cdot r^j$$

are the real part of ramified meromorphic functions on $(\Delta, 0)$ up to bounded functions, we have $g_{i,\mathbf{w}}(z) = \sum \beta_{i,j,\mathbf{w}} z^j \in z^{-1}\mathbb{C}[z^{-1}]$ such that $\varphi_{\mathbf{w}}^*(h_i^{Q_0}) = \mathrm{Re}(g_{i,\mathbf{w}}(z))$. Because $\mathrm{Re}(\beta_{i,j,\mathbf{w}} e^{\sqrt{-1}\theta}) = \alpha_{i,j}(\theta, \mathbf{w})$, we can

deduce that $\beta_{i,j,\mathbf{w}}$ are real analytic functions of \mathbf{w} . We set $g_i(z, \mathbf{w}) := g_{i,\mathbf{w}}(z)$ and $\beta_{i,j}(\mathbf{w}) := \beta_{i,j,\mathbf{w}}$. We have $h_i^{Q_0} = \varpi^*(g_i)$ on \mathcal{U}_{Q_0} , and that the coefficients $\beta_{i,j}$ are \mathbb{C} -valued functions which are real analytic with respect to \mathbf{w} .

Take any $\mathbf{w} \in H_{P_0}$, $\mathbf{b} \in \mathbb{C}^{n-1}$ and $M \in \mathbb{Z}_{>0}$. If $\epsilon > 0$ is sufficiently small, we have the holomorphic map $\varphi_{M,\mathbf{w},\mathbf{b}} : \Delta_\epsilon \rightarrow X$ given by $\varphi_{M,\mathbf{w},\mathbf{b}}(\zeta) = (\zeta^M, \mathbf{w} + \zeta\mathbf{b})$. Let $\varpi_0 : \tilde{\Delta} \rightarrow \Delta$ be the real blowing up of Δ along 0. Let $\tilde{\varphi}_{M,\mathbf{w},\mathbf{b}} : \tilde{\Delta}_\epsilon \rightarrow \tilde{X}(H)$ be the induced map. We have a small sector S in $\tilde{\Delta}_\epsilon$ such that $\tilde{\varphi}_{M,\mathbf{w},\mathbf{b}}(\tilde{S}) \subset \mathcal{U}_{Q_0}$. The growth order of $\pi^{-1}(\mathbb{C}_S) \otimes \mathbf{E}(\tilde{\varphi}_{M,\mathbf{w},\mathbf{b}} \circ \varpi)^{-1}K$ is controlled by the functions $\tilde{\varphi}_{M,\mathbf{w},\mathbf{b}}^* h_i^{Q_0} = \varpi_0^* \varphi_{M,\mathbf{w},\mathbf{b}}^* \text{Re}(g_i)$. We use the polar coordinate $\zeta = re^{\sqrt{-1}\theta}$ on $\tilde{\Delta}_\epsilon$. By taking the Taylor expansion at $r = 0$, we obtain the following expansion:

$$\begin{aligned} \varpi_0^* \varphi_{M,\mathbf{w},\mathbf{b}}^* (\text{Re } g_i) &= \text{Re} \left(\sum_{-N_1 \leq j < 0} \beta_{ij}(\mathbf{w} + \zeta\mathbf{b}) \zeta^{Mj} \right) \\ &= \sum_{-N_1 M \leq k < 0} \gamma_{i,k,(M,\mathbf{w},\mathbf{b})}(\theta) r^k + \sum_{k \geq 0} \gamma_{i,k,(M,\mathbf{w},\mathbf{b})}(\theta) r^k =: A_{i,(M,\mathbf{w},\mathbf{b}),-} + A_{i,(M,\mathbf{w},\mathbf{b}),+}. \end{aligned} \quad (17)$$

We have $f \in \text{Irr}(\mathbf{E}\varphi_{M,\mathbf{w},\mathbf{b}}^{-1}K)$ such that $\text{Re}(f) - A_{i,(M,\mathbf{w},\mathbf{b}),-}$ is bounded on S . Hence, for $k < 0$, $\gamma_{i,k,(M,\mathbf{w},\mathbf{b})}(\theta)r^k$ is of the form $\text{Re}(\delta_{i,k,(M,\mathbf{w},\mathbf{b})}\zeta^k)$, where $\delta_{i,k,(M,\mathbf{w},\mathbf{b})} \in \mathbb{C}$.

For any fixed i , let $j_0 = j_0(i)$ be the maximum such that $\beta_{i,-j} \neq 0$. Then, we have

$$\begin{aligned} \text{Re}(\beta_{i,-j_0}(\mathbf{w} + \zeta\mathbf{b})\zeta^{-Nj_0}) &= \\ \text{Re}(\beta_{i,-j_0}(\mathbf{w})\zeta^{-Nj_0} + \sum_k \bar{b}_k \partial_{\bar{w}_k} \beta_{i,-j_0}(\mathbf{w}) \bar{\zeta} \zeta^{-Nj_0} + \sum_k b_k \partial_{w_k} \beta_{i,-j_0}(\mathbf{w}) \zeta \zeta^{-Nj_0} + \dots) \end{aligned} \quad (18)$$

By looking at the term $\gamma_{i,-Nj_0+1,(M,\mathbf{w},\mathbf{b})}(\theta, \mathbf{w})r^{-Nj_0+1}$, we can deduce $\partial_{\bar{w}_k} \beta_{i,-j_0}(\mathbf{w}) = 0$ for any $\mathbf{w} \in H_{P_0}$. Hence, we obtain that $\beta_{i,-j_0}$ is holomorphic. By a descending inductive argument, we obtain that $\beta_{i,-j}$ are holomorphic for any $j > 0$.

Let $\varphi : \Delta \rightarrow X$ be any holomorphic map such that $\varphi(0) \in H$ and $\varphi(\Delta \setminus \{0\}) \subset X \setminus H$. Let $\varpi_0 : \tilde{\Delta} \rightarrow \Delta$ be the oriented blowing up along 0. We have the induced map $\tilde{\varphi} : \tilde{\Delta} \rightarrow \tilde{X}(H)$. We have a sector $S = \{(\theta, r) \in \tilde{\Delta} \mid \theta_1 < \theta < \theta_2, 0 \leq r < \epsilon'\}$ such that $\tilde{\varphi}(S) \subset \mathcal{U}_{Q_0}$. The growth order of $\pi^{-1}(\mathbb{C}_S) \otimes \mathbf{E}(\varpi_0 \circ \tilde{\varphi})^{-1}(K)$ is controlled by $\tilde{\varphi}^* h_i^{Q_0}$. Hence, we obtain that $\text{Irr}(\mathbf{E}\varphi^{-1}K) = \{\varphi^* g_i\}$, which is clearly compatible with the multiplicity. Thus, we obtain that K satisfies **(A)** at P_0 . \blacksquare

4.2 Meromorphic functions and subanalytic functions

4.2.1 Extension of meromorphic functions

Let X , H and $\varpi : \tilde{X}(H) \rightarrow X$ be as in §4.1. Let ν be a real analytic function on H such that $d\nu \neq 0$ at $H_0 := \nu^{-1}(0)$. We set $H_\pm := \{P \in H \mid \pm \nu(P) > 0\}$. We assume that $H = H_0 \times]-1, 1[$ as a real analytic manifold, where the projection to $] -1, 1[$ is given by ν .

Let f be an analytic function defined on $X_+ := \Delta \times H_+$ of the form

$$f(z, P) = \text{Re} \left(\sum_{0 < j \leq N_1} \alpha_j(P) z^{-j} \right),$$

where α_j are holomorphic functions defined on H_+ .

Let $H_1 \subset H_0$ be a relatively compact open subset. We use the polar coordinate $z = re^{\sqrt{-1}\theta}$. For a positive integer m and positive numbers θ_1 , c_1 and ϵ_1 , we set

$$B(H_1, m, c_1, \theta_1, \epsilon_1) := \{(r, \theta, \nu) \mid \nu > 0, 0 < r < c_1 \nu^m, |\theta| < \theta_1\} \times H_1 \subset X \setminus H$$

Its closure in $\tilde{X}(H)$ is denoted by $\overline{B}(H_1, m, c_1, \theta_1, \epsilon_1)$. Suppose that we are given H_1 , m , c_1 , θ_1 and ϵ_1 as above, and a function g on $B(H_1, m, c_1, \theta_1, \epsilon_1)$ as a power series

$$g = \sum_{i \geq -N_1} \sum_{j \geq -N_2} (r\nu^{-m})^{i/\rho} \nu^{j/\rho} \beta_{ij}(\theta, \nu),$$

where v varies on H_1 . We assume the following.

- $f - g$ is bounded around any $(0, \theta, \nu, v) \in \overline{B}(H_1, m, c_1, \theta_1, \epsilon_1)$ such that $\nu > 0$.

Lemma 4.2 *The function $\sum_{0 < j \leq N_1} \alpha_j z^{-j}$ is extended to a meromorphic function on a neighbourhood of $\Delta \times \{\nu \geq 0\}$.*

Proof Fix a point $P_0 \in H_0$. By shrinking H_0 around P_0 , we may assume to have holomorphic functions w_1, \dots, w_{n-2} on H such that $F|_{H_0} : H_0 \rightarrow \mathbb{C}^{n-2}$ is submersive, where $F : H \rightarrow \mathbb{C}^{n-2}$ is given by (w_1, \dots, w_{n-2}) . By shrinking H_0 around P_0 , we may assume to have a \mathbb{C} -valued real analytic function η on H such that (i) $\eta|_{F^{-1}(P')}$ are holomorphic for any $P' \in \mathbb{C}^{n-2}$, (ii) $H_0 = \{\text{Re } \eta = 0\}$. We may assume that $(\eta, w_1, \dots, w_{n-2})$ gives an open embedding of H to \mathbb{C}^{n-1} . We may assume that $\nu = \text{Re}(\eta)$. We set $\mu := \text{Im}(\eta)$. Tuples (w_1, \dots, w_{n-2}) are denoted by \mathbf{w} .

By comparing the power expansions with respect to r , we obtain the following equality:

$$f = \sum_{-N_1 \leq i < 0} \sum_{j \geq -N_2} (r\nu^{-m})^{i/\rho} \nu^{j/\rho} \beta_{ij}(\theta, \mu, \mathbf{w}). \quad (19)$$

Moreover, for any fixed (θ, \mathbf{w}) , the function $\sum_j \nu^{-mi/\rho} \nu^{j/\rho} \beta_{ij}(\theta, \mu, \mathbf{w})$ is harmonic with respect to $\partial_\nu^2 + \partial_\mu^2$. So, we obtain that the only non-negative integer powers of ν appear in (19), and that α_j are extended to holomorphic functions on a neighbourhood of $\{\nu \geq 0\}$. \blacksquare

4.2.2 Boundedness

We impose the following additional condition to g .

(P) Let $\varphi : \Delta \rightarrow X$ be any holomorphic map such that $\varphi(0) \in H_0$. Then, for each connected component \mathcal{C} of $\varphi^{-1}B(H_1, m, c_1, \theta_1, \epsilon_1)$, we have $\kappa \in \mathbb{Z}_{>0}$ and $h \in \mathbb{C}[\zeta^{-1/\kappa}]$ such that $\varphi^*g - \text{Re}(h)$ is bounded on \mathcal{C} .

Lemma 4.3 *$f - g$ is bounded on $B(H_1, m, c_1, \theta_1, \epsilon_1)$.*

Proof We use the notation in the proof of Lemma 4.2. It is enough to prove $\beta_{i,j}(\theta, v) = 0$ for $i \geq 0$ and $j < 0$. We assume that $\sum_{i \geq 0} \sum_{0 > j \geq -N_2} (r\nu^{-m})^{i/\rho} \nu^{j/\rho} \beta_{ij}(\theta, \mu, \mathbf{w}) \neq 0$, and we shall derive a contradiction. We may assume that there exist $i_0 \geq 0$ and (μ_0, \mathbf{w}_0) such that $\beta_{i_0, -N_2}(\theta, \mu_0, \mathbf{w}_0)$ is not constantly 0. We may assume $\mu_0 = 0$. We have the expansion $\beta_{i,j}(\theta, \mu, \mathbf{w}_0) = \sum_{k \geq 0} \beta_{i,j,k}(\theta, \mathbf{w}_0) \mu^k$.

We set

$$g_0 := \sum_{i \geq 0} \sum_{j \geq -N_2} (r\nu^{-m})^{i/\rho} \nu^{j/\rho} \beta_{ij}(\theta, \mu, \mathbf{w}).$$

The condition (P) is satisfied for g_0 .

We consider the holomorphic map $\varphi : \{(\zeta, a) \mid |\zeta| < \epsilon, |1 - a| < \epsilon\} \rightarrow X$ given by $\varphi(\zeta, a) = (\zeta^m, a\zeta, \mathbf{w}_0)$, where we use the coordinate (z, η, \mathbf{w}) on X . Note that η is holomorphic on $F^{-1}(\mathbf{w}_0)$. On the domain $\varphi^{-1}(B(H_1, m, c_1, \theta_1, \epsilon_1))$, we have

$$\begin{aligned} \varphi^*(g_0)(\zeta) &= \sum_{i \geq 0} \sum_{j \geq -N_2} \sum_{k \geq 0} |\zeta|^{mi/\rho} \text{Re}(a\zeta)^{(-mi+j)/\rho} \text{Im}(a\zeta)^k \beta_{i,j,k}(\arg(\zeta), \mathbf{w}_0) \\ &= \sum_{i \geq 0} \sum_{j \geq -N_2} \sum_{k \geq 0} |\zeta|^{j/\rho+k} |a|^{(-mi+j)/\rho+k} \cos(\arg(a\zeta))^{(-mi+j)/\rho} \sin(\arg(a\zeta))^k \beta_{i,j,k}(m \arg(\zeta), \mathbf{w}_0). \end{aligned} \quad (20)$$

By the argument in the proof of Lemma 4.1, we obtain that the coefficient of $|\zeta|^{-N_2/\rho}$ is harmonic with respect to a , i.e., the function

$$\begin{aligned} \sum_{i \geq 0} |a|^{(-mi-N_2)/\rho} \cos(\arg(a\zeta))^{(-mi-N_2)/\rho} \beta_{i, -N_2, 0}(m \arg(\zeta), \mathbf{w}_0) = \\ \sum_{i \geq 0} \text{Re}(a\zeta)^{(-mi-N_2)/\rho} |\zeta|^{(mi+N_2)/\rho} \beta_{i, -N_2, 0}(m \arg(\zeta), \mathbf{w}_0) \end{aligned} \quad (21)$$

is harmonic with respect to a . It implies $\beta_{i, -N_2, 0}(m \arg(\zeta), \mathbf{w}_0) = 0$ for any $i \geq 0$, which contradicts with the assumption. Thus, we obtain Lemma 4.3. \blacksquare

4.3 Extension along subsets with real codimension three

4.3.1 Preliminary

Let X , H , ν and H_0 , H_\pm be as in §4.2.1. For any $P \in H$, let X_P be a small neighbourhood of P in X , and we set $H_P := H \cap X_P$, $H_{0,P} := H_0 \cap H_P$ and $H_{\pm,P} := H_\pm \cap H_P$. For a positive integer m and positive numbers c_1 , θ_1 and ϵ_1 , we set

$$B_\pm(H_P, m, c_1, \theta_1, \epsilon_1) := \{(r, \theta, \nu) \mid \pm \nu > 0, 0 < r < c_1(\pm \nu)^m, |\theta| < \theta_1\} \times H_{0,P} \subset X \setminus H.$$

Let $K \in \mathbf{E}_\otimes^b(IC_{\mathbf{X}(H)})$. Suppose that K satisfies the condition **(GA)** at $P \in H \setminus H_0$. For simplicity, we assume that the ramified sets of irregular values \mathcal{I}_P are contained in $z^{-1}\mathcal{O}_{H,P}[z^{-1}]$, where $\mathcal{O}_{H,P}$ denote the local ring at P . We also impose the following condition along H_0 .

Assumption 4.4 *Suppose that we are given good sets of irregular values $\mathcal{I}_{P,\kappa} \subset z^{-1}\mathcal{O}_{H,P}[z^{-1}]$ ($\kappa = \pm$) with a multiplicity function $\mathbf{m}_{P,\kappa} : \mathcal{I}_{P,\kappa} \rightarrow \mathbb{Z}_{\geq 0}$ for each $P \in H_0$ such that the following holds for a neighbourhood H_P and some $m(P), c_1(P), \theta_1(P), \epsilon_1(P)$:*

- *Let $\varphi : \Delta \rightarrow X$ be a holomorphic map such that $\varphi^{-1}(B_\kappa(H_P, m(P), c_1(P), \theta_1(P), \epsilon_1(P)))$ contains a sector of Δ^* . Then, $\text{Irr}(\mathbf{E}\varphi^{-1}K) = \varphi^*\mathcal{I}_\kappa$ compatible with the multiplicity.*

Lemma 4.5 *We have $\mathcal{I}_{P,+} = \mathcal{I}_{P,-}$ and $\mathbf{m}_{P,+} = \mathbf{m}_{P,-}$.*

Proof It is enough to consider the case $\dim_{\mathbb{C}} H = 1$. We may assume to have a holomorphic coordinate w on H such that $w(P) = 0$. Suppose that we have $f \in \mathcal{I}_{P,+} \setminus \mathcal{I}_{P,-}$, and we shall derive a contradiction. Let $\mathcal{I}_{P,-} = \{g_1, \dots, g_\ell\}$. We have

$$(f - g_j)(z, \eta, \mathbf{w}) = \sum_{i=1}^{N(j)} a_{j,i}(\eta, \mathbf{w}) z^{-i},$$

where $a_{j,N(j)}$ are not constantly 0. We have $a_{j,i}(w) = w^{\ell(j,i)} \cdot b_{j,i}(w)$, where $b_{j,i}(0) \neq 0$. We have $m(j) \in \mathbb{Z}_{>0}$ such that

$$-mN(j) + \ell(j, N(j)) < -mi + \ell(j, i)$$

for any i such that $a_{j,i} \neq 0$. We set $m_0 := \max\{m(j)\}$.

We take $m \geq m_0$. Let $\varphi_1 : \Delta_\epsilon \rightarrow X$ be given by $\varphi_1(\zeta) = (\zeta^m, \zeta)$. Then, by our choice of m , we have $\varphi_1^*(f - g_j) \neq 0$ in $\mathcal{O}_\Delta(*0)/\mathcal{O}_\Delta$. If m is sufficiently large, then both $\varphi^{-1}(B_\kappa(H_P, m(P), c_1(P), \theta_1(P), \epsilon_1(P)))$ ($\kappa = +, -$) contains a sector of Δ^* . Hence, we have arrived at a contradiction, i.e., $\mathcal{I}_{P,+} \subset \mathcal{I}_{P,-}$.

Similarly, we obtain that $\mathcal{I}_{P,-} \subset \mathcal{I}_{P,+}$. We can also compare the multiplicity functions $\mathbf{m}_{P,+}$ and $\mathbf{m}_{P,-}$ in a similar way. ■

We denote $\mathcal{I}_{P,\pm}$ and $\mathbf{m}_{P,\pm}$ by \mathcal{I}_P and \mathbf{m}_P , respectively.

4.3.2 Extension

Suppose that we are given a holomorphic coordinate (x_1, \dots, x_{n-1}) of H , by which we regard H as an open subset of \mathbb{C}^{n-1} . We also regard X as an open subset of \mathbb{C}^n .

Let $P \in H_0$ and $\mathbf{a}_0 \in \mathbb{C}^{n-1}$. Let $A(\mathbf{a}_0)$ be a neighbourhood of a point \mathbf{a}_0 in \mathbb{C}^{n-1} . Let $m > m(P)$. Let $\Phi_{P,\mathbf{a}_0} : \Delta_\epsilon \times A(\mathbf{a}_0) \times H_{0,P} \rightarrow \mathbb{C}^n$ be given by $\Phi_{P,\mathbf{a}_0}(\xi, \mathbf{a}, \mathbf{x}) = (\xi^m, \mathbf{x} + \xi \mathbf{a})$. We set $Y_{P,\mathbf{a}_0} := \Phi_{P,\mathbf{a}_0}^{-1}(X)$ and $J_{P,\mathbf{a}_0} := \Phi_{P,\mathbf{a}_0}^{-1}(H) = \{0\} \times (A(\mathbf{a}_0) \times H_{0,P})$. We set $\mathbf{Y}_{P,\mathbf{a}_0}(J_{P,\mathbf{a}_0}) := (Y_{P,\mathbf{a}_0} \setminus J_{P,\mathbf{a}_0}, Y_{P,\mathbf{a}_0})$. We have $\mathbf{E}\Phi_{P,\mathbf{a}_0}^{-1}(K)$ in $\mathbf{E}_{\mathbb{R}-c}^b(IC_{\mathbf{Y}_{P,\mathbf{a}_0}(J_{P,\mathbf{a}_0})})$. Let $\varpi_{Y_{P,\mathbf{a}_0}} : \tilde{Y}_{P,\mathbf{a}_0}(J_{P,\mathbf{a}_0}) \rightarrow Y_{P,\mathbf{a}_0}$ denote the oriented real blowing up.

Assumption 4.6 *For any $P \in H_0$, we have $\mathbf{a}(P) \in \mathbb{C}^{n-1}$ and a subanalytic subset $\mathcal{Z} \subset \varpi_{Y_{P,\mathbf{a}(P)}}^{-1}(J_{P,\mathbf{a}(P)})$ with $\dim_{\mathbb{R}} \mathcal{Z} \leq \dim_{\mathbb{R}} J_{P,\mathbf{a}(P)}$ such that the following holds:*

- *\mathcal{Z} is horizontal with respect to $\varpi_{Y_{P,\mathbf{a}(P)}}$.*

- For any $Q \in \varpi_{Y_{P,\mathbf{a}(P)}}^{-1}(J_{P,\mathbf{a}(P)}) \setminus \mathcal{Z}$, we have a neighbourhood \mathcal{U}_Q of Q in $\tilde{Y}_{P,\mathbf{a}(P)}(J_{P,\mathbf{a}(P)})$ such that

$$\pi^{-1}(\mathbb{C}_{\mathcal{U}_Q}) \otimes \mathbf{E}\Phi_{P,\mathbf{a}(P)}^{-1}(K) = \bigoplus_{i=1}^m \mathbb{C}^{\mathbf{E}} \otimes^+ \mathbb{C}_{t \geq h_{Q,i}}$$

for a tuple of ramified analytic functions $h_{Q,i}$ on \mathcal{U}_Q .

We remark the following.

Lemma 4.7 *Hence, $\{h_{Q,i}\}$ in Assumption 4.6 is equal to $\{\operatorname{Re} \Phi_{P,\mathbf{a}(P)}^*(f) \mid f \in \mathcal{I}_P\}$ in $\overline{\operatorname{Sub}}(\mathcal{U}_Q^\circ, \mathcal{U}_Q)$ compatible with the multiplicity.*

Proof For each $\mathbf{x} \in H_{0,P}$, we have a real subspace $T(\mathbf{x}) \subset \mathbb{C}^{n-1}$ with $\dim_{\mathbb{R}} T(\mathbf{x}) = 2n-3$ such that if $\mathbf{a} \notin T(\mathbf{x})$ then $|\nu(\mathbf{a}\xi + \mathbf{x})| \sim b(\mathbf{a}, \mathbf{x})|\xi|$ for $b(\mathbf{a}, \mathbf{x}) > 0$. For such $(\mathbf{a}, \mathbf{x}) \in J_{P,\mathbf{a}(P)}$, let us consider the holomorphic map $\varphi(\zeta) = (\zeta^m, \mathbf{a}\zeta + \mathbf{x})$. Then, $\varphi^{-1}(B_{\pm}(H_P, m(P), c_1(P), \theta_1(P), \epsilon_1(P)))$ contains a sector. Hence, we obtain that $\{h_{Q,i}\}$ is equal to $\{\operatorname{Re} \Phi_{P,\mathbf{a}(P)}^*(f) \mid f \in \mathcal{I}_P\}$ in $\overline{\operatorname{Sub}}(\mathcal{U}_Q^\circ, \mathcal{U}_Q)$ compatible with the multiplicity if $Q \in \varpi_{Y_{P,\mathbf{a}(P)}}^{-1}(\mathbf{a}, \mathbf{x})$ for such (\mathbf{a}, \mathbf{x}) . We obtain the claim for general Q by using the continuity because the functions $h_{Q,i}$ are ramified analytic functions. \blacksquare

Proposition 4.8 *Let $\varphi : \Delta \rightarrow X$ be any holomorphic map such that $\varphi(\Delta \setminus 0) \subset X \setminus H$ and $\varphi(0) \in H_0$. Then, we have $\operatorname{Irr}(\mathbf{E}\varphi^{-1}(K)) = \varphi^* \mathcal{I}_{\varphi(0)}$ compatible with the multiplicity.*

Proof If H_P is sufficiently small, we have a C^∞ -map $q : H_P \rightarrow H_{0,P}$ whose restriction to $H_{0,P}$ is the identity.

Lemma 4.9 *Let $g : \Delta \rightarrow \mathbb{R}_{\geq 0}$ be a non-constant real analytic function. Let $m_1 < m_2$ be positive integers. Let $\theta_1 < \theta_2$. Let ℓ be a positive integer. We set*

$$T_1 := \{\zeta \mid \theta_1 < \arg(\zeta) < \theta_2, |\zeta|^\ell < g(\zeta)^{m_1}\}, \quad T_2 := \{\zeta \mid g(\zeta)^{m_2} < |\zeta|^\ell\}$$

Then, T_1 contains a sector of $(\Delta, 0)$, or T_2 contains a neighbourhood of 0.

Proof Let $\xi = x + \sqrt{-1}y$ be the real coordinate. We have a positive integer k such that $g = \sum_{i+j \geq k} a_{i,j} x^i y^j$, and at least one of $a_{i,j}$ ($i+j = k$) is not 0.

If $k > \ell/m_2$, then T_2 clearly contains a neighbourhood of 0. Let us consider the case $k \leq \ell/m_2 < \ell/m_1$. By considering a coordinate change from ζ to $\beta\zeta$, we may assume $a_{k,0} \neq 0$, and $\theta_1 < 0 < \theta_2$. If we take sufficiently small $\epsilon_i > 0$ ($i = 1, 2$), then g and x^k are mutually bounded on $S = \{(x, y) \mid 0 < x < \epsilon_1, |y| < \epsilon_2\}$. Moreover, S is contained $\{\zeta \mid \theta_1 < \arg(\zeta) < \theta_2\}$, and $|\zeta|$ and x are mutually bounded on S . Then, the claim is clear. \blacksquare

Lemma 4.10 *Let $g_2 : \Delta \rightarrow \mathbb{R}_{\geq 0}$ be a real analytic function. Suppose that $g_2(\zeta) \leq |\zeta|^\ell$ for a positive integer ℓ . Let $\psi : [0, \epsilon] \rightarrow \Delta$ be a real analytic curve such that $\psi(0) = 0$. Suppose that $g_2 \circ \psi(t) = 0$ for any t . Then, for any $\delta > 0$, we have a sector S_δ and $0 < \epsilon' < \epsilon$ such that (i) S_δ contains $\psi([0, \epsilon'])$, (ii) $g_2 \leq \delta|\zeta|^\ell$ on S_δ .*

Proof We have $k > 0$ such that $g_2 = \sum_{i+j \geq k} a_{i,j} x^i y^j$ where at least one of $a_{i,j}$ ($i+j = k$) is not 0. We have $k \geq \ell$. It is enough to study the case $k = \ell$.

Let us consider the case $\psi(t) = t$. We have $a_{k,0} = 0$. Hence, if ρ_1 is sufficiently small, then on $\{|y| < \rho_1 x\}$, we have $g_2 \leq \delta|\zeta|^\ell$.

Let us consider the general case. We have the holomorphic map $\psi_{\mathbb{C}} : \Delta_\epsilon \rightarrow \Delta$ whose restriction to $\{0 \leq t\}$ is equal to ψ . We have a sector S in the upper $(\Delta_\epsilon, 0)$ on which we have $g_2 \circ \psi \leq \delta|\psi^*(\zeta^\ell)|$. Because $\psi_{\mathbb{C}}(S)$ contains a small sector, we obtain the claim of the lemma. \blacksquare

Take any $P \in H_0$. Let $\varphi : \Delta \rightarrow X_P$ be a holomorphic map such that $\varphi(0) \in H_P$ and $\varphi(\Delta \setminus \{0\}) \subset X_P \setminus H_P$. It is described as $(\varphi_z(\zeta), \varphi_{\mathbf{x}}(\zeta))$. By Lemma 4.9, one of the following holds.

- $\varphi^{-1}(B_{\pm}(H_P, c_1(P), \theta_1(P), \epsilon_1(P)))$ contains a sector.
- $T_3 = \{|\varphi_z(\zeta)| > \nu(\varphi_{\mathbf{x}}(\zeta))^m\}$ contains a neighbourhood of 0.

Suppose that the second case occurs. We may assume that $\varphi_z(\zeta) = \zeta^{m_\ell}$. Then, we have $\nu(\varphi_{\mathbf{x}}(\zeta)) < |\zeta|^\ell$ on T_3 . We have $\nu(\varphi_{\mathbf{x}}(\zeta) - \mathbf{a}(P)\zeta^\ell) < C|\zeta|^\ell$.

Let $\psi_{\mathbf{x}} : T_3 \rightarrow H_0$ be given by $\psi_{\mathbf{x}}(\zeta) := q(\varphi_{\mathbf{x}}(\zeta) - \mathbf{a}(P)\zeta^\ell)$. We have the one dimensional analytic subset C of Δ such that $0 \in C$ and that $\varphi_{\mathbf{x}}(\zeta) - \mathbf{a}(P)\zeta^\ell \in H_0$ for $\zeta \in C$. Then, by Lemma 4.10, we can take a sector S and a function $\psi_{\mathbf{a}} : S \rightarrow A(P, \mathbf{a}(P))$ such that $\varphi_{\mathbf{x}}(\zeta) - q(\varphi_{\mathbf{x}}(\zeta) - \mathbf{a}(P)\zeta^\ell) = \psi_{\mathbf{a}}(\zeta)\zeta^\ell$. We obtain a real analytic map $\psi : S \rightarrow Y_{P, \mathbf{a}(P)}$ given by $\psi(\zeta) = (\zeta^\ell, \psi_{\mathbf{a}}(\zeta), \psi_{\mathbf{x}}(\zeta))$ such that $\Phi_{P, \mathbf{a}(P)} \circ \psi = \varphi$. Let \bar{S} denote the closure of S in the oriented real blowing up $\tilde{\Delta}$ of Δ at 0. The map ψ is extended to the real analytic map $\bar{S} \rightarrow \tilde{Y}_{P, \mathbf{a}(P)}(J_{P, \mathbf{a}(P)})$. Then, by Lemma 4.7, we have $\text{Irr}(\mathbb{E}\varphi^{-1}K) = \{\varphi^* \text{Re}(f) \mid f \in \mathcal{I}_P\}$ compatible with the multiplicity. Thus, we obtain the claim of the proposition. \blacksquare

We obtain the following from Proposition 3.32, under the assumptions.

Corollary 4.11 *We have a good meromorphic flat bundle V on (X, H) with an isomorphism $\text{DR}_{X(H)}^{\mathbb{E}}(V)[-n] \simeq K$.* \blacksquare

4.3.3 Remark on Assumption 4.6

Let X be an n -dimensional complex manifold. Let H be a normal crossing hypersurface of X . Let $H^{[2]}$ be the singular locus of H . Let $C \subset H$ be a closed subanalytic subset with $\dim_{\mathbb{R}} C = 2n - 3$ such that $H^{[2]} \subset C$.

Let $K \in \mathbb{E}_{\odot}^b(I\mathbb{C}_{X(H)})$. Let m be a positive integer. We prepare a notation. Let P be any smooth point of $C \setminus H^{[2]}$. When we take a small holomorphic coordinate neighbourhood $(X_P; z, x_1, \dots, x_{n-1})$ of X around P such that $H_P = H \cap X_P = \{z = 0\}$. Set $C_P := X_P \cap C$. We regard X_P as an open subset of \mathbb{C}^n .

Let $\Phi_P : \mathbb{C} \times \mathbb{C}^{n-1} \times C_P \rightarrow \mathbb{C}^n$ be given by $\Phi_P(\xi, \mathbf{a}, \mathbf{x}) = (\xi, \mathbf{x} + \xi^m \mathbf{a})$. We set $Y_P := \Phi_P^{-1}(X_P)$ and $J_P := \Phi_P^{-1}(H_P) = \{0\} \times \mathbb{C}^{n-1} \times C_P$. Set $\mathbf{Y}_P(J_P) = (Y_P \setminus J_P, Y_P)$. Set $\varpi_{Y_P} : \tilde{Y}_P(J_P) \rightarrow Y_P$ be the oriented real blowing up. We have $\mathbb{E}\Phi_P^{-1}(K) \in \mathbb{E}_{\mathbb{R}-c}^b(I\mathbb{C}_{\mathbf{Y}_P(J_P)})$.

Lemma 4.12 *We have a closed subanalytic subset $Z \subset C$ with $\dim_{\mathbb{R}} Z = 2n - 4$ such that the following holds.*

- Z contains $H^{[2]}$ and the singular locus of C .
- Take any $P \in C \setminus Z$. We use the above notation. Then, we have a closed subanalytic subset $W_P \subset \varpi_P^{-1}(J_P)$ such that (i) W_P is horizontal over J_P , (ii) for any $Q \in \varpi_{Y_P}^{-1}(J_P) \setminus W_P$, we have a neighbourhood \mathcal{U}_Q of Q in $\tilde{Y}_P(J_P)$ such that $\pi^{-1}(\mathbb{C}_{\mathcal{U}_Q}) \otimes \mathbb{E}\Phi_P^{-1}(K)$ is controlled by ramified analytic functions. In other words, the assumption 4.6 is satisfied around P .

Proof It is enough to consider the case $X = \Delta_z \times \Delta_{\mathbf{x}}^{n-1}$, $H = \{z = 0\} \cup \bigcup_{j=1}^{\ell} \{x_j = 0\}$ and C is contained in $\{z = 0\}$. We set $H^{(0)} := \{z = 0\}$.

We take a $(2n - 3)$ -dimensional real analytic manifold M with a proper map $\rho : M \rightarrow X$ such that $\rho(M) = C$. Note that we have a closed real analytic subset $W_M \subset M$ with $\dim W_M \leq 2n - 4$ such that (i) $\rho^{-1}\rho(W_M) = W_M$, (ii) $M \setminus W_M \simeq C \setminus \Phi(W_M)$, (iii) W_M contains the pull back of $H^{[2]}$ and the singular locus of C by ρ .

Let $\Phi : \mathbb{C} \times \mathbb{C}^{n-1} \times M \rightarrow \mathbb{C}^n$ be given by $\Phi(\xi, \mathbf{a}, y) = (\xi, \rho(y) + \xi^m \mathbf{a})$. We set $Y := \Phi^{-1}(X)$ and $J := \Phi^{-1}(H^{(0)}) = \{0\} \times \mathbb{C}^{n-1} \times M$. We also set $\mathcal{H} := \Phi^{-1}(H)$.

Let $\varpi_Y : \tilde{Y}(J) \rightarrow Y$ be the oriented real blowing up. Let $j : (Y \setminus \mathcal{H}, \tilde{Y}(J)) \rightarrow \tilde{Y}(J) = (\tilde{Y}(J), \tilde{Y}(J))$ be the inclusion of the bordered spaces. Let $\tilde{K} := \mathbb{E}_{j!} \mathbb{E}\Phi^{-1}K$. We have the filtration by open subanalytic subsets $\tilde{Y}(J) = \tilde{Y}(J)^{(0)} \supset \tilde{Y}(J)^{(1)} \supset \dots$ such that (i) $\tilde{Y}(J)^{(j)} \setminus \tilde{Y}(J)^{(j+1)}$ are submanifolds of codimension j , (ii) for each connected component \mathcal{C} of $\tilde{Y}(J)^{(j)} \setminus \tilde{Y}(J)^{(j+1)}$, we have subanalytic functions $h_1^{\mathcal{C}}, \dots, h_{k(\mathcal{C})}^{\mathcal{C}}$ on $(\mathcal{C}, \tilde{Y}(J))$

such that $\pi^{-1}(\mathbb{C}_{\mathcal{C}}) \otimes \tilde{K} = \bigoplus \mathbb{C}^{\mathbb{E}} \otimes^+ \mathbb{C}_{t \geq h_{\ell}^{\mathcal{C}}}$. Applying Lemma 2.17 to the components \mathcal{C} with maximal dimension and the functions $h_{\ell}^{\mathcal{C}}$, we obtain a closed subanalytic subset $W_1 \subset \varpi_Y^{-1}(J)$ with $\dim_{\mathbb{R}} W_1 \leq \dim_{\mathbb{R}} J$ such that the following holds.

- For any $Q \in \varpi_Y^{-1}(J) \setminus W_1$, we have a neighbourhood \mathcal{U}_Q of Q in $\tilde{Y}(J)$ and a ramified analytic functions h_i^Q such that $\pi^{-1}(\mathbb{C}_{\mathcal{U}_Q}) \otimes \tilde{K} = \bigoplus \mathbb{C}^{\mathbb{E}} \otimes^+ \mathbb{C}_{t \geq h_i^Q}$.

We have a closed subanalytic subset $W_2 \subset J$ with $\dim_{\mathbb{R}} W_2 < \dim_{\mathbb{R}} J$ such that $W_1 \setminus \varpi_Y^{-1}(W_2)$ is horizontal over J . We have a closed subanalytic subset $W_3 \subset C$ with $\dim_{\mathbb{R}} W_3 \leq 2n-4$ such that (i) W_3 contains $\rho(W_M)$, $H^{[2]}$ and the singular locus of C , (ii) for any $P \in C \setminus W_3$, we have $\dim_{\mathbb{R}}(W_2 \cap \Phi^{-1}(P)) < \dim_{\mathbb{R}} \Phi^{-1}(P)$. Then, the claim of the lemma follows. \blacksquare

4.4 Construction outside of subsets with real codimension four

Let X be an n -dimensional complex manifold. Let H be a simple normal crossing hypersurface of X .

Proposition 4.13 *For any $K \in \mathbb{E}_{\odot}^b(I\mathbb{C}_{\mathbf{X}(H)})$, we have a closed subanalytic subset $Z \subset H$ with $\dim_{\mathbb{R}} Z \leq 2n-4$, a good meromorphic flat bundle V on $(X', H') := (X \setminus Z, H \setminus Z)$ and an isomorphism $\mathrm{DR}_{\mathbf{X}'(H')}^{\mathbb{E}}(V)[-d_X] \simeq K|_{X' \setminus H'}$ in $\mathbb{E}^b(I\mathbb{C}_{\mathbf{X}'(H')})$.*

4.4.1 Preliminary

Let $\varpi : \tilde{X}(H) \rightarrow X$ be the oriented real blowing up. We have the bordered space $\tilde{\mathbf{X}}(H) := (X \setminus H, \tilde{X}(H))$, and ϖ gives an isomorphism of bordered spaces $\tilde{\mathbf{X}}(H) \rightarrow \mathbf{X}(H)$. We have $\mathbb{E}\varpi^{-1}K \in \mathbb{E}_{\mathbb{R}-c}^b(\tilde{\mathbf{X}}(H))$. We have a filtration $\tilde{X}(H) = \tilde{X}(H)^{(0)} \supset \tilde{X}(H)^{(1)} \supset \dots$ for $\mathbb{E}\varpi^{-1}K$. For any connected component \mathcal{C} of $\tilde{X}(H) \setminus \tilde{X}(H)^{(1)}$, we have subanalytic functions $h_1^{\mathcal{C}}, \dots, h_{\ell}^{\mathcal{C}}$ on $(\mathcal{C}, \tilde{X}(D))$ such that $\pi^{-1}(\mathbb{C}_{\mathcal{C}}) \otimes K \simeq \bigoplus_j \mathbb{C}_{\tilde{X}(H)}^{\mathbb{E}} \otimes_{\mathbb{C}_{t \geq h_j^{\mathcal{C}}}}^+ \mathbb{C}_{t \geq h_j^{\mathcal{C}}}$. By Lemma 2.17 and Lemma 4.12, we have the following.

Lemma 4.14 *We have $(2n-3)$ -dimensional closed subanalytic subset $Z_0 \subset H$ and a $(2n-2)$ -dimensional closed subanalytic subset $R_0 \subset \partial\tilde{X}(H)$ with the following property.*

- Z_0 contains the singular locus $H^{[2]}$ of H .
- R_0 contains $\varpi^{-1}(H^{[2]})$. Let W be the closure of $\tilde{X}(H)^{(1)} \setminus \partial\tilde{X}(H)$ in $\tilde{X}(H)$. Then, the intersection $W \cap \partial\tilde{X}(H)$ is also contained in R_0 .
- The induced map $R_0 \setminus \varpi^{-1}(Z_0) \rightarrow H$ is relatively 0-dimensional.
- Let Q be any point of $\partial\tilde{X}(H) \setminus R_0$. Then, we have a neighbourhood \mathcal{U}_Q of Q in $\tilde{X}(H)$ and analytic functions h_1^Q, \dots, h_{ℓ}^Q on $\mathcal{U}_Q^{\circ} = \mathcal{U}_Q \setminus \varpi^{-1}(H)$ which are ramified analytic around Q , such that $\pi^{-1}(\mathbb{C}_{\mathcal{U}_Q}) \otimes K = \bigoplus_j \mathbb{C}_{\tilde{X}(H)}^{\mathbb{E}} \otimes_{\mathbb{C}_{t \geq h_j^Q}}^+ \mathbb{C}_{t \geq h_j^Q}$.

We have $(2n-3)$ -dimensional closed subanalytic subset $R_1 \subset Z_0 \times_H \partial\tilde{X}(H)$ and $(2n-4)$ -dimensional closed subanalytic subset $Z_1 \subset Z_0$ with the following property.

- Z_1 contains the singular locus of Z_0 .
- $R_1 \setminus \varpi^{-1}(Z_1) \rightarrow H$ is relatively 0-dimensional.
- Take $P_0 \in Z_0 \setminus Z_1$ and a relatively compact open neighbourhood H_{P_0} of P_0 in H . Let Q be any point of $\varpi^{-1}(Z_0 \setminus Z_1) \setminus R_1$ such that $P := \varpi(Q) \in H_{P_0}$. We have a real analytic coordinate neighbourhood $(\mathcal{N}; y_1, \dots, y_{2n-2})$ of H around P such that $Z_0 \cap \mathcal{N} = \{y_{2n-2} = 0\}$. We take real numbers $\theta_1 < \theta_2$ such that the interval $[\theta_1, \theta_2]$ is a small neighbourhood of Q in $\varpi^{-1}(P) \setminus R_1$. Then, we have a positive integer ρ , a positive integer m , a positive number $C > 0$, connected components $\mathcal{C}(Q, \pm)$ of $\tilde{X}(H) \setminus \tilde{X}(H)^{(1)}$ such that

$$\mathcal{U}_{\pm} = \{(y_1, \dots, y_{2n-2}, \theta, r) \mid (y_1, \dots, y_{2n-2}) \in \mathcal{N}, \theta_1 < \theta < \theta_2, 0 < r < C(\pm y_{2n-2})^m\} \subset \mathcal{C}(Q, \pm)$$

and that the restriction of $h_p^{\mathcal{C}(Q, \pm)}$ to \mathcal{U}_{\pm} are expressed as

$$h_p^{\mathcal{C}(Q, \pm)} = \sum_{i \geq -N_1} \sum_{j \geq -N_2} \alpha_{(\pm, p), i, j}(y_1, \dots, y_{2n-3}, \theta) \cdot y_{2n-2}^{i/\rho} \cdot (y_{2n-2}^{-m} r)^{j/\rho}.$$

Here $\alpha_{(\pm, p), i, j}$ are analytic functions.

Moreover, the assumption 4.6 is satisfied around P_0 . \blacksquare

4.4.2 Proof of Proposition 4.13

Let $P \in H$. We have a neighbourhood X_P of the form $\Delta^\ell \times X_{P,0}$ such that $H_P = H \cap X_P = \bigcup_{i=1}^\ell \{z_i = 0\}$.

Let \mathcal{U} be a connected component of $\partial\tilde{X}_P(H_P) \setminus R_0$. By enlarging R_i and Z_i , we may assume that \mathcal{U} is simply connected. For any $Q \in \partial\tilde{X}_P(H_P) \setminus R_0$, we have functions h_1^Q, \dots, h_ℓ^Q as in the condition of Lemma 4.14. We consider the case $Q \in \varpi^{-1}(\{z_1 = 0\})$. Let $z_1 = r_1 e^{\sqrt{-1}\theta_1}$ be the polar coordinate. We have ramified analytic functions $h_i^Q = \sum_{j \geq -N(i)} h_{i,j}^Q r_1^{j/\rho}$ around Q . We have the analytic functions $h_{i,j}^Q$ ($j < 0$) on a neighbourhood of Q in $\partial\tilde{X}_P(H_P)$. By varying Q in \mathcal{U} , we obtain analytic functions $h_{i,j}^\mathcal{U}$ on \mathcal{U} for $i = 1, \dots, \ell$ and for $j = -1, \dots, -N(i)$. Set $h_i^\mathcal{U} = \sum h_{i,j}^\mathcal{U} r_1^{j/\rho}$.

Lemma 4.15 $h_{i,j}^\mathcal{U}$ are subanalytic functions on $(\mathcal{U}, \tilde{X}(H))$.

Proof We have a connected component \mathcal{C} of $\tilde{X}(H) \setminus \tilde{X}(H)^{(1)}$ such that $\mathcal{U} \subset \overline{\mathcal{C}}$. We may assume that $h_i^\mathcal{C} - h_i^\mathcal{U}$ are bounded on any neighbourhood of $Q \in \mathcal{U}$ in $\tilde{X}(H)$. Hence, $h_{i,-N(i)}^\mathcal{U}$ is described as the restriction of $r_1^{N(i)} h_i^\mathcal{C}$ to $r_1 = 0$. Hence, we obtain that $h_{i,-N(i)}^\mathcal{C}$ is subanalytic on $(\mathcal{U}, \partial\tilde{X}(H))$. Suppose we have already known that $h_{i,j}^\mathcal{C}$ are subanalytic on $(\mathcal{U}, \partial\tilde{X}(H))$ for $-N(i) \leq j \leq -m-1$ for $m < 0$. Then, $h_{i,-m}^\mathcal{U}$ is described as the restriction of $r_1^{m/\rho} (h_i^\mathcal{C} - \sum_{j < -m} r_1^{-j/\rho} h_{i,j}^\mathcal{U})$ to $r_1 = 0$. Hence, we obtain the claim Lemma 4.15. \blacksquare

Take any point $P_1 \in (H_P \setminus Z_0) \cap \{z_1 = 0\}$. By Lemma 4.1, we have the set of ramified irregular values \mathcal{I}_{P_1} at P_1 . Each $f_i^{P_1} \in \mathcal{I}_{P_1}$ is of the form $f_i^{P_1} = \sum_{j=1}^{N(i,j)} f_{i,j}^{P_1} z^{-j/\rho}$. We may assume that each connected component U of $(H_P \setminus Z_0) \cap \{z_1 = 0\}$ is simply connected. We have $f_i^U = \sum f_{i,j}^U z^{-j/\rho}$ whose restriction to a neighbourhood of $P_1 \in U$ is equal to $f_i^{P_1}$. We obtain holomorphic functions $f_{i,j}^U$ on U .

Lemma 4.16 $\operatorname{Re} f_{i,j}^U$ and $\operatorname{Im} f_{i,j}^U$ are subanalytic functions on (U, H) .

Proof Set $H_{P,j} := \{z_j = 0\}$ and $H_{P,1}^\circ := H_{P,1} \setminus \left(\bigcup_{j=2}^\ell H_{P,j} \right)$. We may assume to have $U \subset H_{P,1}^\circ$. We have $\varpi^{-1}(H_{P,1}^\circ) \simeq S^1 \times H_{P,1}^\circ$. For any $\theta \in S^1$, the set $\{\theta\} \times H_{P,1}^\circ$ is a locally closed subanalytic subset in $\partial\tilde{X}(H)$. We have a 0-dimensional closed subset $Z_{S^1} \subset S^1$ such that $\dim\left((\theta \times H_{P,1}^\circ) \cap R_0\right) \leq 2n-3$ for any $\theta \in S^1 \setminus Z_{S^1}$. Let \mathcal{B}_θ be any connected component of $(\theta \times H_{P,1}^\circ) \setminus R_0$. By the previous lemma, we obtain that $\operatorname{Re}(f_{i,j}^U e^{j\sqrt{-1}\theta})$ are subanalytic functions on $(\mathcal{B}_\theta \cap (\theta \times U), \partial\tilde{X}(H))$. Hence, we obtain that $\operatorname{Re}(f_{i,j}^U e^{j\sqrt{-1}\theta})$ are subanalytic functions on $(\theta \times U, \partial\tilde{X}(H))$. Then, we obtain the claim of the lemma. \blacksquare

Lemma 4.17 Let P_1 be any point of $Z_0 \setminus Z_1$. If $P_1 \in \overline{U}$, $f_{i,j}^U$ are holomorphic on a neighbourhood of P_1 .

Proof It follows from Lemma 4.2. \blacksquare

We have closed subanalytic subset $Z_{\overline{U}} \subset \overline{U}$ with $\dim_{\mathbb{R}} Z_{\overline{U}} \leq 2n-4$ such that the following holds.

- $Z_{\overline{U}} \supset Z_1 \cap \overline{U}$.
- For any $P_1 \in \overline{U} \setminus Z_{\overline{U}}$, $\{f_i^U\}$ gives a good set of ramified irregular values at P_1 .

For each $P_1 \in U \setminus Z_{\overline{U}}$, we take a small neighbourhood X_{P_1} in X_P , and we set $H_{P_1} := H_P \cap X_{P_1}$. Proposition 3.32, we have a good meromorphic flat bundle V_{P_1} on (X_{P_1}, H_{P_1}) whose set of ramified irregular values are given by $\{f_i^U\}$, and an isomorphism

$$\operatorname{DR}_{X_{P_1}(H_{P_1})}^E(V_{P_1})[-d_X] \simeq K_{|X_{P_1}}.$$

Let $C(U) := (H \setminus U) \cup Z_{\overline{U}}$. We set $X_U := X_P \setminus C(U)$ and $H_U := H_P \setminus C(U)$. By gluing them X_{P_1} and V_{P_1} for $P_1 \in U \setminus Z_{\overline{U}}$, we obtain a good meromorphic flat bundle V_U on (X_U, H_U) and an isomorphism

$$\operatorname{DR}_{X_U(H_U)}^E(V_U)[-d_X] \simeq K_{|X_U}.$$

Let $Z_P^{(1)}$ denote the union of $Z_{\overline{U}}$ for connected components U of $(H_P \setminus Z_0) \cap \{z_1 = 0\}$. We set $X_P^{(1)} := X_P \setminus (Z_P^{(1)} \cup \bigcup_{j=2}^{\ell} \{z_j = 0\})$, and $H_P^{(1)} := H_P \cap X_P^{(1)}$.

Take any $P_1 \in H_P^{(1)} \cap Z_0$. Note that Z_0 is smooth around P_1 . Let U_1, U_2 denote the connected components of $(H_P \setminus Z_0) \cap \{z_1 = 0\}$ such that $P_1 \in \overline{U}_i$. Note that $\{f_i^{U_a}\}$ ($a = 1, 2$) are good at P . We take a small neighbourhood X_{P_1} of P_1 in $X_P^{(1)}$, and set $H_{P_1} := X_{P_1} \cap H_P^{(1)}$. Then, we have good meromorphic flat bundles $V_{P_1}^{(a)}$ ($a = 1, 2$) such that the restriction of $V_{P_1}^{(a)}$ to $X_{P_1} \cap X_{U_a}$ are isomorphic to the restriction of V_{U_a} to $X_{P_1} \cap X_{U_a}$. By Corollary 4.11, we obtain $V_{P_1}^{(1)} \simeq V_{P_1}^{(2)}$. Hence, we have a good meromorphic flat bundle V_{P_1} on (X_{P_1}, H_{P_1}) and an isomorphism

$$\mathrm{DR}_{X_{P_1}(H_{P_1})}^E(V_{P_1})[-d_X] \simeq K_{|X_{P_1}}.$$

Hence, by gluing V_U for connected components U , and V_P for $P \in H_P^{(1)} \cap Z_0$, we obtain a good meromorphic flat bundle $V^{(1)}$ on $(X_P^{(1)}, H_P^{(1)})$ and an isomorphism

$$\mathrm{DR}_{X_P^{(1)}(H_P^{(1)})}^E(V_P^{(1)})[-d_X] \simeq K_{|X_P^{(1)}}.$$

Similarly, for each $i = 2, \dots, \ell$, we have the following:

- a closed subanalytic subset $Z_P^{(i)}$ of $H_P \cap \{z_i = 0\}$, for which $(Z_1 \cap \{z_i = 0\}) \subset Z_P^{(i)}$.
- a good meromorphic flat bundle $V^{(i)}$ on $(X_P^{(i)}, H_P^{(i)})$, where

$$X_P^{(i)} := X_P \setminus \left(Z_P^{(i)} \cup \bigcup_{\substack{1 \leq j \leq \ell \\ j \neq i}} \{z_j = 0\} \right), \quad H_P^{(i)} = X_P^{(i)} \cap H_P.$$

- an isomorphism $\mathrm{DR}_{X_P^{(i)}(H_P^{(i)})}^E(V_P^{(i)})[-d_X] \simeq K_{|X_P^{(i)}}$.

We set $Z_P := \bigcup_{i=1}^{\ell} Z_P^{(i)}$. By gluing $V_P^{(i)}$ ($i = 1, \dots, \ell$), we obtain a good meromorphic flat bundle V_P on $(X_P', H_P') := (X_P \setminus Z_P, H_P \setminus Z_P)$, and an isomorphism $\mathrm{DR}_{X_P'(H_P')}^E(V_P)[-d_X] \simeq K_{|X_P'}$.

We have a subsets $S \subset X$ and a covering $X = \bigcup_{P \in S} X_P$ which is a locally finite covering. Let Z denote the union of the closures of Z_P ($P \in S$). By gluing V_P ($P \in S$), we obtain a good meromorphic flat bundle V and the desired isomorphism. Thus, the proof of Proposition 4.13 is finished. \blacksquare

5 Sequence of blowings up at points

5.1 Some notation

5.1.1 The basic case

Let \mathbb{K} denote \mathbb{R} or \mathbb{C} . Let (x, y) be the standard coordinate of \mathbb{K}^2 . Let $p : \mathrm{Bl}_{(0,0)} \mathbb{K}^2 \rightarrow \mathbb{K}^2$ be the blowing up at $(0, 0)$. It is given as $\{((x, y), [u : v]) \in \mathbb{K}^2 \times \mathbb{P}^1(\mathbb{K}) \mid xv - uy = 0\}$. We have the points $P_+ := ((0, 0), [1 : 0])$ and $P_- := ((0, 0), [0 : 1])$. We have the coordinate neighbourhood $(U_+, (u_+, v_+))$ given by $(u_+, v_+) = (x, y/x)$ around P_+ , and $(U_-, (u_-, v_-))$ given by $(u_-, v_-) = (x/y, y)$ around P_- . We identify $U_{\pm} \simeq \mathbb{K}^2$ by the coordinate. The restriction of p to U_{\pm} are denoted by p_{\pm} . The morphisms p_{\pm} are described as $p_+(x_1, y_1) = (x_1, x_1 y_1)$ and $p_-(x_1, y_1) = (x_1 y_1, y_1)$ with respect to the standard coordinate (x_1, y_1) on \mathbb{K}^2 .

For any $\alpha \neq 0$, we have the point $P_{\alpha} = (\alpha, 0) \in U_-$. We have the coordinate neighbourhood $(U_{\alpha}, u_{\alpha}, v_{\alpha})$ of P_{α} given by $U_{\alpha} := U_-$ and $(u_{\alpha}, v_{\alpha}) := (u_- - \alpha, v_-)$. For the coordinate $(U_{\alpha}, u_{\alpha}, v_{\alpha})$, P_{α} is described as $(0, 0)$. For the coordinate (U_+, u_+, v_+) , P_{α} is denoted as $(0, \alpha^{-1})$.

5.1.2 Sequence of blowing up

We set $\mathfrak{P}(\mathbb{K}) := (\mathbb{K} \setminus \{0\}) \cup \{+, -\}$. Let $\boldsymbol{\eta} = (\eta_1, \dots, \eta_\ell)$ be an element in $\mathfrak{P}(\mathbb{K})^\ell$ for some $\ell \geq 1$. We shall construct a sequence of spaces $X_{\boldsymbol{\eta}}^{(i)}$ ($i = 0, \dots, \ell$) with points $P_{\boldsymbol{\eta}}^{(i)}$, maps $p^{(i)} : (X_{\boldsymbol{\eta}}^{(i)}, P_{\boldsymbol{\eta}}^{(i)}) \longrightarrow (X_{\boldsymbol{\eta}}^{(i-1)}, P_{\boldsymbol{\eta}}^{(i-1)})$, and coordinate neighbourhoods $(U_{\boldsymbol{\eta}}^{(i)}, u_{\boldsymbol{\eta}}^{(i)}, v_{\boldsymbol{\eta}}^{(i)})$ around $P_{\boldsymbol{\eta}}^{(i)}$, given as follows.

We set $X_{\boldsymbol{\eta}}^{(0)} := \mathbb{K}^2$ and $P_{\boldsymbol{\eta}}^{(0)} = (0, 0)$. The coordinate neighbourhood $(U_{\boldsymbol{\eta}}^{(0)}, u_{\boldsymbol{\eta}}^{(0)}, v_{\boldsymbol{\eta}}^{(0)})$ is given by (\mathbb{K}^2, x, y) . Let $p_{\boldsymbol{\eta}}^{(1)} : X_{\boldsymbol{\eta}}^{(1)} \longrightarrow X_{\boldsymbol{\eta}}^{(0)}$ be the blowing up at $P_{\boldsymbol{\eta}}^{(0)}$. We have the natural isomorphism $X_{\boldsymbol{\eta}}^{(1)} \simeq \text{Bl}_{(0,0)} \mathbb{K}^2$. Let $P_{\boldsymbol{\eta}}^{(1)} \in X_{\boldsymbol{\eta}}^{(1)}$ be the point corresponding to $P_{\eta_1} \in \text{Bl}_{(0,0)} \mathbb{K}^2$. The coordinate neighbourhood $(U_{\boldsymbol{\eta}}^{(1)}, u_{\boldsymbol{\eta}}^{(1)}, v_{\boldsymbol{\eta}}^{(1)})$ is given as $(U_{\eta_1}, u_{\eta_1}, v_{\eta_1})$.

Suppose that we already have $X_{\boldsymbol{\eta}}^{(i)}$, $P_{\boldsymbol{\eta}}^{(i)}$ and $(U_{\boldsymbol{\eta}}^{(i)}, u_{\boldsymbol{\eta}}^{(i)}, v_{\boldsymbol{\eta}}^{(i)})$. Let $p_{\boldsymbol{\eta}}^{(i+1)} : X_{\boldsymbol{\eta}}^{(i+1)} \longrightarrow X_{\boldsymbol{\eta}}^{(i)}$ be the blowing up at $P_{\boldsymbol{\eta}}^{(i)}$. The inclusion $U_{\boldsymbol{\eta}}^{(i)} \longrightarrow X_{\boldsymbol{\eta}}^{(i)}$ induces $\text{Bl}_{P_{\boldsymbol{\eta}}^{(i)}} U_{\boldsymbol{\eta}}^{(i)} \longrightarrow X_{\boldsymbol{\eta}}^{(i+1)}$. We have the isomorphism $\text{Bl}_{P_{\boldsymbol{\eta}}^{(i)}} U_{\boldsymbol{\eta}}^{(i)} \simeq \text{Bl}_{(0,0)} \mathbb{K}^2$ induced by the coordinate system $(u_{\boldsymbol{\eta}}^{(i)}, v_{\boldsymbol{\eta}}^{(i)})$. Let $P_{\boldsymbol{\eta}}^{(i+1)}$ be the point corresponding to $P_{\eta_{i+1}}$. Let $(U_{\boldsymbol{\eta}}^{(i+1)}, u_{\boldsymbol{\eta}}^{(i+1)}, v_{\boldsymbol{\eta}}^{(i+1)})$ be the coordinate neighbourhood around $P_{\boldsymbol{\eta}}^{(i+1)}$ corresponding to $(U_{\eta_{i+1}}, u_{\eta_{i+1}}, v_{\eta_{i+1}})$.

We set $\text{Bl}_{\boldsymbol{\eta}} \mathbb{K}^2 := X_{\boldsymbol{\eta}}^{(\ell)}$, $P_{\boldsymbol{\eta}} := P_{\boldsymbol{\eta}}^{(\ell)}$ and $(U_{\boldsymbol{\eta}}, u_{\boldsymbol{\eta}}, v_{\boldsymbol{\eta}}) := (U_{\boldsymbol{\eta}}^{(\ell)}, u_{\boldsymbol{\eta}}^{(\ell)}, v_{\boldsymbol{\eta}}^{(\ell)})$.

5.1.3 Infinite sequence

For any $\boldsymbol{\eta} = (\eta_1, \eta_2, \dots) \in \mathfrak{P}(\mathbb{K})^\infty$, we set $\boldsymbol{\eta}_k := (\eta_1, \dots, \eta_k)$. We obtain the sequence of the spaces $\text{Bl}_{\boldsymbol{\eta}_k} \mathbb{K}^2$ with the base point $P_{\boldsymbol{\eta}_k}$ and the coordinate neighbourhoods $(U_{\boldsymbol{\eta}_k}, u_{\boldsymbol{\eta}_k}, v_{\boldsymbol{\eta}_k})$. We have the naturally induced morphism $p_{\boldsymbol{\eta}}^{(k)} : (\text{Bl}_{\boldsymbol{\eta}_k} \mathbb{K}^2, P_{\boldsymbol{\eta}_k}) \longrightarrow (\text{Bl}_{\boldsymbol{\eta}_{k-1}} \mathbb{K}^2, P_{\boldsymbol{\eta}_{k-1}})$.

5.2 Sequences of blowings up at cross points

5.2.1 Explicit descriptions

Let us consider the case $\boldsymbol{\eta} \in \{+, -\}^\ell$. Let $\psi_{\boldsymbol{\eta}}^{(i)} : X_{\boldsymbol{\eta}}^{(i)} \longrightarrow \mathbb{K}^2$ be the induced map. Let us describe the restriction $(\psi_{\boldsymbol{\eta}}^{(i)})|_{U_{\boldsymbol{\eta}}^{(i)}} \longrightarrow \mathbb{K}^2$ under the identification $\mathbb{K}^2 \simeq U_{\boldsymbol{\eta}}^{(i)}$ by $(x_1, y_1) = (u_{\boldsymbol{\eta}}^{(i)}, v_{\boldsymbol{\eta}}^{(i)})$. We set

$$A_+ := \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, \quad A_- := \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}. \quad (22)$$

We set $A_{\boldsymbol{\eta}}^{(i)} = A_{\eta_1} A_{\eta_2} \cdots A_{\eta_i}$. Let $\alpha^{(i)}(\boldsymbol{\eta})$, $\beta^{(i)}(\boldsymbol{\eta})$, $\gamma^{(i)}(\boldsymbol{\eta})$ and $\delta^{(i)}(\boldsymbol{\eta})$ be the components of $A_{\boldsymbol{\eta}}^{(i)}$:

$$A_{\boldsymbol{\eta}}^{(i)} = \begin{pmatrix} \alpha^{(i)}(\boldsymbol{\eta}) & \beta^{(i)}(\boldsymbol{\eta}) \\ \gamma^{(i)}(\boldsymbol{\eta}) & \delta^{(i)}(\boldsymbol{\eta}) \end{pmatrix}.$$

Then, we have $\psi_{\boldsymbol{\eta}}^{(i)}(x_1, y_1) = (x_1^{\alpha^{(i)}(\boldsymbol{\eta})} y_1^{\beta^{(i)}(\boldsymbol{\eta})}, x_1^{\gamma^{(i)}(\boldsymbol{\eta})} y_1^{\delta^{(i)}(\boldsymbol{\eta})})$.

Let $\psi_{\boldsymbol{\eta}}$ denote the map $\psi_{\boldsymbol{\eta}}^{(\ell)} : \text{Bl}_{\boldsymbol{\eta}} \mathbb{K}^2 \longrightarrow \mathbb{K}^2$. We set $A_{\boldsymbol{\eta}}^{(\ell)} := A_{\boldsymbol{\eta}}$, $\alpha^{(\ell)}(\boldsymbol{\eta}) = \alpha(\boldsymbol{\eta})$, $\beta^{(\ell)}(\boldsymbol{\eta}) = \beta(\boldsymbol{\eta})$, $\gamma^{(\ell)}(\boldsymbol{\eta}) = \gamma(\boldsymbol{\eta})$, and $\delta^{(\ell)}(\boldsymbol{\eta}) = \delta(\boldsymbol{\eta})$. The restriction $(\psi_{\boldsymbol{\eta}})|_{U_{\boldsymbol{\eta}}} : U_{\boldsymbol{\eta}} \longrightarrow \mathbb{K}^2$ is denoted as

$$\psi_{\boldsymbol{\eta}}(u_{\boldsymbol{\eta}}, v_{\boldsymbol{\eta}}) = (u_{\boldsymbol{\eta}}^{\alpha(\boldsymbol{\eta})} v_{\boldsymbol{\eta}}^{\beta(\boldsymbol{\eta})}, u_{\boldsymbol{\eta}}^{\gamma(\boldsymbol{\eta})} v_{\boldsymbol{\eta}}^{\delta(\boldsymbol{\eta})}).$$

5.2.2 Polar decompositions

We have the natural inclusion $P_{\boldsymbol{\eta}} \in \text{Bl}_{\boldsymbol{\eta}} \mathbb{R}^2 \subset \text{Bl}_{\boldsymbol{\eta}} \mathbb{C}^2$. We have the coordinate neighbourhood $(U_{\boldsymbol{\eta}, \mathbb{K}}, u_{\boldsymbol{\eta}, \mathbb{K}}, v_{\boldsymbol{\eta}, \mathbb{K}})$ of $P_{\boldsymbol{\eta}}$ in $\text{Bl}_{\boldsymbol{\eta}} \mathbb{K}^2$, where the subscript \mathbb{K} is included for distinction. The coordinate functions $u_{\boldsymbol{\eta}, \mathbb{R}}$ and $v_{\boldsymbol{\eta}, \mathbb{R}}$ are the restriction of $u_{\boldsymbol{\eta}, \mathbb{C}}$ and $v_{\boldsymbol{\eta}, \mathbb{C}}$, respectively.

Let $\text{Bl}_{\boldsymbol{\eta}}(\mathbb{R}_{\geq 0}^2)$ denote the strict transform of $\mathbb{R}_{\geq 0}^2 \subset \mathbb{R}^2$ with respect to the morphism $\psi_{\boldsymbol{\eta}} : \text{Bl}_{\boldsymbol{\eta}}(\mathbb{R}^2) \longrightarrow \mathbb{R}^2$. Namely, $\text{Bl}_{\boldsymbol{\eta}}(\mathbb{R}_{\geq 0}^2)$ is the closure of $\psi_{\boldsymbol{\eta}}^{-1}(\mathbb{R}_{\geq 0}^2)$ in $\text{Bl}_{\boldsymbol{\eta}}(\mathbb{R}^2)$. We have the induced morphism $\psi_{\boldsymbol{\eta}, \mathbb{R}_{\geq 0}} : \text{Bl}_{\boldsymbol{\eta}}(\mathbb{R}_{\geq 0}^2) \longrightarrow \mathbb{R}_{\geq 0}^2$. We have $U_{\boldsymbol{\eta}, \mathbb{R}} \cap \text{Bl}_{\boldsymbol{\eta}}(\mathbb{R}_{\geq 0}^2) = \{u_{\boldsymbol{\eta}, \mathbb{R}} \geq 0, v_{\boldsymbol{\eta}, \mathbb{R}} \geq 0\}$.

Let $H := \{x = 0\} \cup \{y = 0\}$ in \mathbb{C}^2 . We set $H_\eta := \psi_\eta^{-1}(H) \subset \text{Bl}_\eta \mathbb{C}^2$. We have the oriented real blowing up $\varpi : \widetilde{\mathbb{C}^2}(H) \rightarrow \mathbb{C}^2$ and $\varpi_\eta : \widetilde{\text{Bl}_\eta \mathbb{C}^2}(H_\eta) \rightarrow \text{Bl}_\eta \mathbb{C}^2$. We have the induced morphism $\widetilde{\psi}_{\eta, \mathbb{C}} : \widetilde{\text{Bl}_\eta \mathbb{C}^2}(H_\eta) \rightarrow \widetilde{\mathbb{C}^2}(H)$.

By the polar decompositions of the coordinate functions $u_{\eta, \mathbb{C}} = r_{\eta, 1} e^{\sqrt{-1}\theta_{\eta, 1}}$ and $v_{\eta, \mathbb{C}} = r_{\eta, 2} e^{\sqrt{-1}\theta_{\eta, 2}}$, we obtain $\varpi_\eta^{-1}(U_{\eta, \mathbb{C}}) \simeq \mathbb{R}_{\geq 0}^2 \times (S^1)^2$. We also have the natural identification $\widetilde{\mathbb{C}^2}(H) \simeq \mathbb{R}_{\geq 0}^2 \times (S^1)^2$. The following is clear by the expression of the morphisms $\psi_{\eta, \mathbb{C}}$.

Lemma 5.1 *The restriction of $\widetilde{\psi}_{\eta, \mathbb{C}}$ to $\varpi_\eta^{-1}(U_{\eta, \mathbb{C}})$ is identified with the product of the following morphisms:*

- the induced morphism $\psi_{\eta, \mathbb{R}_{\geq 0}} : \text{Bl}_\eta(\mathbb{R}_{\geq 0}^2) \cap U_{\eta, \mathbb{R}} \rightarrow \mathbb{R}_{\geq 0}^2$,
- the isomorphism $(S^1)^2 \rightarrow (S^1)^2$ given by $(\theta_{\eta, 1}, \theta_{\eta, 2}) \mapsto (\alpha(\eta)\theta_{\eta, 1} + \beta(\eta)\theta_{\eta, 2}, \gamma(\eta)\theta_{\eta, 1} + \delta(\eta)\theta_{\eta, 2})$. ■

5.3 Sequence of blowings up of mixed type and the induced family of paths

5.3.1 The induced family of curves

Let $\mathbf{Y} = (\eta_1, \omega_1, \dots, \eta_k, \omega_k) \in \coprod_{\ell \geq 0} \mathfrak{P}^\ell$. Here, $\eta_j \in \{+, -\}^{\ell(j)}$ and $\omega_j \in \mathbb{C}^*$. For any $m \leq k$, we set $\mathbf{Y}_m := (\eta_1, \omega_1, \dots, \eta_m, \omega_m)$. We have the morphism $\psi_{\mathbf{Y}_m} : \text{Bl}_{\mathbf{Y}_m} \mathbb{C}^2 \rightarrow \mathbb{C}^2$, the point $P_{\mathbf{Y}_m} \in \text{Bl}_{\mathbf{Y}_m} \mathbb{C}^2$ and the coordinate neighbourhood $(U_{\mathbf{Y}_m}, u_{\mathbf{Y}_m}, v_{\mathbf{Y}_m})$. Instead of an explicit description of $\psi_{\mathbf{Y}_m} : U_{\mathbf{Y}_m} \rightarrow \mathbb{C}^2$ with respect to $(x_1, y_1) := (u_{\mathbf{Y}_m}, v_{\mathbf{Y}_m})$ and (x, y) , we study a description of an induced family of curves.

Take a small positive number $\epsilon > 0$. We have the family of holomorphic curves

$$\varphi_m : (\mathbb{C} \setminus \{-\omega_m\}) \times \Delta_\epsilon \rightarrow U_{\mathbf{Y}_m}$$

given by $\varphi_m(a, \zeta) := (a, \zeta)$. As the composite with $\psi_{\mathbf{Y}_m}$, we obtain a family of holomorphic curves

$$\psi_{\mathbf{Y}_m} \circ \varphi_m : (\mathbb{C} \setminus \{-\omega_m\}) \times \Delta_\epsilon \rightarrow \mathbb{C}^2.$$

Let $(\psi_{\mathbf{Y}_m} \circ \varphi_m)_x(a, \zeta)$ and $(\psi_{\mathbf{Y}_m} \circ \varphi_m)_y(a, \zeta)$ denote the x -component and y -component of $\psi_{\mathbf{Y}_m} \circ \varphi_m(a, \zeta)$:

$$\psi_{\mathbf{Y}_m} \circ \varphi_m(a, \zeta) = ((\psi_{\mathbf{Y}_m} \circ \varphi_m)_x(a, \zeta), (\psi_{\mathbf{Y}_m} \circ \varphi_m)_y(a, \zeta)).$$

5.3.2 Change of parametrization

To obtain a more convenient description of the family of curves $\psi_{\mathbf{Y}_m} \circ \varphi_m$, we change a parametrization of the curves. For $i = 1, \dots, m$, let $(\alpha_i, \beta_i, \gamma_i, \delta_i) \in \mathbb{Z}^4$ be determined as follows:

$$\begin{pmatrix} \alpha_i & \beta_i \\ \gamma_i & \delta_i \end{pmatrix} = A_{\eta_i} \cdot \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}.$$

We set $\delta(i) := \prod_{j=1}^i \delta_j$. We can check the following lemma by a direct computation.

Lemma 5.2 *The function*

$$G_m(a, \zeta) := \zeta^{-\delta(m)} \cdot \prod_{i=1}^m \omega_i^{-\gamma_i \delta(i-1)} \cdot (\psi_{\mathbf{Y}_m} \circ \varphi_m)_y$$

is holomorphic on $(\mathbb{C} \setminus \{-\omega_m\}) \times \Delta_\epsilon$, and $G_m(a, 0) = 1$. ■

We have the map $\mathbb{C}_u \rightarrow \mathbb{C}_y$ given by $u \mapsto u^{\delta(m)}$. We denote the variable u by $y^{1/\delta(m)}$. We fix a $\delta(m)/\delta(i-1)$ -root of $\omega_i^{\gamma_i}$ for each i . They determine a holomorphic map

$$(\psi_{\mathbf{Y}_m} \circ \varphi_m)^{1/\delta(m)} : \mathcal{U}_{m,1} \rightarrow \mathbb{C}_{y^{1/\delta(m)}}$$

such that $\left((\psi_{\mathbf{Y}_m} \circ \varphi_m)^{1/\delta(m)}\right)^{\delta(m)} = \psi_{\mathbf{Y}_m} \circ \varphi_m$, where $\mathcal{U}_{m,1}$ is a small neighbourhood of $(\mathbb{C} \setminus \{-\omega_m\}) \times \{0\}$ in $(\mathbb{C} \setminus \{-\omega_m\}) \times \Delta_\epsilon$.

We have the map $\Phi_m : \mathcal{U}_{m,1} \longrightarrow \mathbb{C}_a \times \mathbb{C}_{y^{1/\delta(m)}}$ given by $\Phi_m(a, \zeta) = \left(a, (\psi_{\mathbf{Y}_m} \circ \varphi_m)^{1/\delta(m)}(a, \zeta)\right)$. By shrinking $\mathcal{U}_{m,1}$, we may assume that Φ_m gives an isomorphism of $\mathcal{U}_{m,1}$ and $\mathcal{U}_{m,2} := \Phi_m(\mathcal{U}_{m,1})$. We obtain the following holomorphic function on $\mathcal{U}_{m,2}$:

$$g_{\mathbf{Y}_m} := (\psi_{\mathbf{Y}_m} \circ \varphi_m)_x \circ \Phi_m^{-1}.$$

We obtain the family of holomorphic curves $F_{\mathbf{Y}_m} := \psi_{\mathbf{Y}_m} \circ \varphi_m \circ \Phi_m^{-1} : \mathcal{U}_{m,2} \longrightarrow \mathbb{C}_x \times \mathbb{C}_y$:

$$F_{\mathbf{Y}_m}(a, y^{1/\delta(m)}) = (g_{\mathbf{Y}_m}(a, y^{1/\delta(m)}), y).$$

In this way, we obtain a collection of families of holomorphic curves $F_{\mathbf{Y}_m}$ ($m = 1, \dots, k$). Let $\mathcal{S}(\mathbf{Y}_m) := \{\eta \in \mathbb{Q} \mid g_{\mathbf{Y}_m, \eta} \neq 0\}$.

5.3.3 Special case: sequences of blowing up at smooth points

Let $\mathbf{Y} = (\boldsymbol{\eta}_1, \omega_1, \dots, \boldsymbol{\eta}_k, \omega_k)$ be as in §5.3.1. In the case $\boldsymbol{\eta}_i \in \{-\}^{\ell(i)}$ for any i , we have a simple description of the family of curves. In this case, it is convenient to regard $-\in \mathfrak{P}$ as 0, i.e., we identify $\{-\} \sqcup \mathbb{C}^*$ with \mathbb{C} in \mathfrak{P} . Let $\boldsymbol{\omega} = (\omega_1, \dots, \omega_m) \in \mathbb{C}^r \subset \mathfrak{P}^r$. We have the map $\psi_{\boldsymbol{\omega}} : \text{Bl}_{\boldsymbol{\omega}} \mathbb{C}^2 \longrightarrow \mathbb{C}^2$. By a direct computation, the map $\psi_{\boldsymbol{\omega}} : (U_{\boldsymbol{\omega}}, u_{\boldsymbol{\omega}}, v_{\boldsymbol{\omega}}) \longrightarrow (\mathbb{C}^2, x, y)$ is described as follows:

$$\psi_{\boldsymbol{\omega}}(u_{\boldsymbol{\omega}}, v_{\boldsymbol{\omega}}) = \left(\sum_{i=1}^{m-1} \omega_i v_{\boldsymbol{\omega}}^i + (\omega_m + u_{\boldsymbol{\omega}}) v_{\boldsymbol{\omega}}^m, v_{\boldsymbol{\omega}}\right).$$

Hence, the family $F_{\mathbf{Y}} : \mathbb{C} \setminus \{-\omega_m\} \times \Delta \longrightarrow \mathbb{C}^2$ is described as follows:

$$F_{\mathbf{Y}}(a, y) = \left(\sum_{i=1}^{m-1} \omega_i y^i + (\omega_m + a) y^m, y\right).$$

5.3.4 Statements

We obtain the following rational numbers

$$\kappa(\mathbf{Y}_m) := \sum_{j=1}^m \frac{\beta_j}{\delta(j)}.$$

We clearly have $\kappa(\mathbf{Y}_{m'}) < \kappa(\mathbf{Y}_m)$ for $m' < m$. We shall prove the following proposition in §5.3.6–§5.3.7.

Proposition 5.3 *If $\eta < \kappa(\mathbf{Y}_m)$, then $g_{\mathbf{Y}_m, \eta}(a)$ are independent of a . The coefficient $g_{\mathbf{Y}_m, \kappa(\mathbf{Y}_m)}(a)$ is an affine function of a . Moreover, for $m < m_1$, the following holds:*

- $g_{\mathbf{Y}_{m_1}, \eta} = g_{\mathbf{Y}_m, \eta}$ for $\eta < \kappa(\mathbf{Y}_m)$. We also have $g_{\mathbf{Y}_{m_1}, \kappa(\mathbf{Y}_m)}(a) = g_{\mathbf{Y}_m, \kappa(\mathbf{Y}_m)}(0)$.

Under some additional assumption, we obtain the following, which will also be proved in §5.3.6–§5.3.7.

Proposition 5.4 *Suppose that there exists a prime \mathfrak{p}_0 such that $\text{ord}_{\mathfrak{p}_0}(\kappa(\mathbf{Y}_m)) < 0$ and $\text{ord}_{\mathfrak{p}_0}(\kappa(\mathbf{Y}_m)) < \text{ord}_{\mathfrak{p}_0}(\kappa(\mathbf{Y}_{m'}))$ for any $m' < m$. Then, the following holds:*

- $g_{\mathbf{Y}_m, \kappa(\mathbf{Y}_m)}(0) \neq 0$.
- For any $\eta \in \mathcal{S}(\mathbf{Y}_m)$ such that $\eta < \kappa(\mathbf{Y}_m)$, we have $\text{ord}_{\mathfrak{p}_0}(\kappa(\mathbf{Y}_m)) < \text{ord}_{\mathfrak{p}_0}(\eta)$.

We give a remark.

Lemma 5.5 *Let $N_0 := \min\{N \mid \kappa(\mathbf{Y}_i) \in \frac{1}{N}\mathbb{Z} \ (i = 1, \dots, m-1)\}$. If $\kappa(\mathbf{Y}_m) \in \frac{1}{N_0}\mathbb{Z}$, then $\delta_m = 1$.*

Proof By the choice of N_0 , $a_i := N_0 \beta_i / \delta(i)$ are integers for any $i \leq m-1$, and we have $\text{g.c.d.}(a_{i_0}, N_0) = 1$ for some $i_0 \leq m-1$. Because $a_{i_0} \delta(i_0) N_0^{-1} \in \mathbb{Z}$, we have $\delta(i_0) N_0^{-1} \in \mathbb{Z}$. It implies that $A = N_0^{-1} \delta(m-1)$ is an integer. We have $\kappa(\mathbf{Y}_m) = \kappa(\mathbf{Y}_{m-1}) + N_0^{-1} A^{-1} \delta_m^{-1} \beta_m$. If $\kappa(\mathbf{Y}_m) \in \frac{1}{N_0}\mathbb{Z}$ we need to have $A^{-1} \delta_m^{-1} \beta_m \in \mathbb{Z}$. Because $\text{g.c.d.}(\delta_m, \beta_m) = 1$, we obtain $\delta_m = 1$. \blacksquare

5.3.5 Limit curve

Let $\mathbf{Y} \in \mathfrak{P}^\infty$. We have the limit $\widehat{\kappa}(\mathbf{Y}) := \lim_{m \rightarrow \infty} \kappa(\mathbf{Y}_m)$ in $\mathbb{R} \cup \{\infty\}$. For any $\eta < \widehat{\kappa}(\mathbf{Y})$, we have the limit $g_{\mathbf{Y},\eta} := \lim_{m \rightarrow \infty} g_{\mathbf{Y}_m,\eta}$ by Proposition 5.3. When $\eta \geq \widehat{\kappa}(\mathbf{Y})$, we formally set $g_{\mathbf{Y},\eta} = 0$. We set $g_{\mathbf{Y}} = \sum g_{\mathbf{Y},\eta} y^\eta$ in $\prod_{\eta \in \mathbb{Q}_{\geq 0}} \mathbb{C} y^\eta$. We set $\mathcal{S}(g_{\mathbf{Y}}) := \{\eta \in \mathbb{Q}_{\geq 0} \mid g_{\mathbf{Y},\eta} \neq 0\}$. For any $\kappa < \widehat{\kappa}(\mathbf{Y})$, the set $\{0 \leq \eta \leq \kappa\} \cap \mathcal{S}(g_{\mathbf{Y}})$ is finite. We call $g_{\mathbf{Y}}(y)$ the limit curve. (See also [7].)

We introduce a condition for infinite sequences.

- We say that the limit curve of \mathbf{Y} is convergent if we have N such that $\mathcal{S}(g_{\mathbf{Y}}) \in \frac{1}{N}\mathbb{Z}$, and the power series $g_{\mathbf{Y}}$ is convergent. Note that the condition implies there exists i_0 such that $\boldsymbol{\eta}_i = (-, \dots, -)^{\ell(i)}$ for any $i \geq i_0$. It also implies that $\widehat{\kappa}(\mathbf{Y}) = \infty$.

Later, we shall divide \mathfrak{P}^∞ into the following three classes.

Class (i): We have $\widehat{\kappa}(\mathbf{Y}) < \infty$.

Class (ii): We have $\widehat{\kappa}(\mathbf{Y}) = \infty$, but the limit curve of \mathbf{Y} is not convergent.

Class (iii): The limit curve of \mathbf{Y} is convergent.

5.3.6 Inductive step

Let us start the proof of Proposition 5.3. For $\boldsymbol{\eta} \in \{+, -\}^\ell$, we set $\boldsymbol{\eta}- = (\boldsymbol{\eta}, -) \in \{+, -\}^{\ell+1}$. Let $\alpha, \beta, \gamma, \delta$ be determined by

$$\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} = A_{\boldsymbol{\eta}-} = A_{\boldsymbol{\eta}} \cdot \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}.$$

Note that $\beta > 0$ and $\delta > 0$. The map $\psi_{\boldsymbol{\eta}-} : (U_{\boldsymbol{\eta}-}, u_{\boldsymbol{\eta}-}, v_{\boldsymbol{\eta}-}) \rightarrow (\mathbb{C}^2, x, y)$ is given by $\psi_{\boldsymbol{\eta}-}(u_{\boldsymbol{\eta}-}, v_{\boldsymbol{\eta}-}) = (u_{\boldsymbol{\eta}-}^\alpha v_{\boldsymbol{\eta}-}^\beta, u_{\boldsymbol{\eta}-}^\gamma v_{\boldsymbol{\eta}-}^\delta)$.

Let \mathcal{U} be a neighbourhood of 0. Let N be a positive integer. Let $\kappa \in \frac{1}{N}\mathbb{Z}_{>0}$. Suppose that we are given a holomorphic function $P : \mathcal{U} \times \Delta_\epsilon \rightarrow \mathbb{C}$ satisfying the following conditions:

- For the expansion $P(a, \sigma) = \sum_{j \geq 0} P_{j/N}(a) \sigma^j$, the coefficients $P_{j/N}(a)$ are independent of a if $j/N < \kappa$. Moreover, $P_0 \neq 0$.

We consider a family of curves $\varphi : \mathcal{U} \times \Delta_\epsilon \rightarrow (U_{\boldsymbol{\eta}-}, u_{\boldsymbol{\eta}-}, v_{\boldsymbol{\eta}-})$ given by

$$\varphi(a, \sigma) = (P(a, \sigma), \sigma^N).$$

We obtain the induced family of curves $\psi_{\boldsymbol{\eta}-} \circ \varphi : \mathcal{U} \times \Delta_\epsilon \rightarrow (\mathbb{C}^2, x, y)$:

$$\psi_{\boldsymbol{\eta}-} \circ \varphi(a, \sigma) = (P(a, \sigma)^\alpha \sigma^{N\beta}, P(a, \sigma)^\gamma \sigma^{N\delta}). \quad (23)$$

We choose $P_0^{\gamma/N\delta} \in \mathbb{C}$, and then $P(a, \sigma)^{\gamma/N\delta}$ is given. Set $v := P(a, \sigma)^{\gamma/N\delta} \sigma$.

We have the map $\Phi : \mathcal{U} \times \Delta_\epsilon \rightarrow \mathcal{U} \times \mathbb{C}$ given by $(a, \sigma) \mapsto (a, v(a, \sigma))$. The restriction of Φ to a small neighbourhood of $\mathcal{U} \times \{0\}$ gives a bi-holomorphic map into $\mathcal{U} \times \mathbb{C}$. We take a smaller neighbourhood \mathcal{U}' of 0 in \mathcal{U} , and $\epsilon' < \epsilon$ appropriately. Then, we have a holomorphic map $Q : \mathcal{U}' \times \Delta_{\epsilon'} \rightarrow \mathbb{C}$ such that the following conditions are satisfied:

- $\sigma = Q(a, v) \cdot v$, i.e., $\Phi^{-1}(a, v) = (a, Q(a, v) \cdot v)$.
- For the expansion $Q(a, v) = \sum_{j \geq 0} Q_{j/N}(a) v^j$, the coefficients $Q_{j/N}(a)$ ($j/N < \kappa$) are determined by P_η ($\eta < \kappa$). In particular, they are independent of a .
- We have $Q_0 = P_0^{-\gamma/N\delta} \neq 0$.
- The function $Q_\kappa(a) + (\gamma/N\delta) P_0^{-1-\gamma\kappa/\delta-\gamma/N\delta} P_\kappa(a)$ is constant.

We can check the following lemma by direct computations.

Lemma 5.6 *Suppose that there exists a prime \mathfrak{p}_0 such that the following holds:*

- *We have $\text{ord}_{\mathfrak{p}_0}(\kappa) < 0$.*
- *If $P_\eta \neq 0$ and $\eta < \kappa$, then $\text{ord}_{\mathfrak{p}_0}(\eta) > \text{ord}_{\mathfrak{p}_0}(\kappa)$.*

Then, we also have the following for Q :

- *If $Q_\eta \neq 0$ and $\eta < \kappa$, then $\text{ord}_{\mathfrak{p}_0}(\eta) > \text{ord}_{\mathfrak{p}_0}(\kappa)$.*
- *We have $Q_\kappa(a) = -(\gamma/N\delta)P_0^{-1-\gamma\kappa/\delta-\gamma/N\delta}P_\kappa(a)$.* ■

We obtain the holomorphic map $R : \mathcal{U} \times \Delta_{\epsilon'} \longrightarrow \mathbb{C}$ as follows:

$$R(a, v) := P(a, vQ(a, v))$$

Lemma 5.7 *For the expansion $R(a, v) = \sum_{j \geq 0} R_{j/N}(a)v^j$, $R_{j/N}(a)$ ($j/N < \kappa$) are determined by P_η ($\eta < \kappa$). In particular, they are independent of a . We have $R_0 = P_0 \neq 0$. Moreover, if the assumption of Lemma 5.6 is satisfied, we have the following:*

- *If $R_\eta \neq 0$ and $\eta < \kappa$, then $\text{ord}_{\mathfrak{p}_0}(\eta) > \text{ord}_{\mathfrak{p}_0}(\kappa)$.*
- *We have $R_\kappa(a) = P_0^{-\gamma\kappa/\delta}P_\kappa(a)$.* ■

We set as follows:

$$S(a, v) := R(a, v)^\alpha Q(a, v)^{N\beta}$$

We have the following description of (23):

$$\psi_{\eta-} \circ \varphi \circ \Phi^{-1}(a, v) = (v^{N\beta}S(a, v), v^{N\delta})$$

Lemma 5.8 *For the expansion $S(a, v) = \sum S_{j/N}(a)v^j$, $S_{j/N}(a)$ ($j/N < \kappa$) depend only on $\{P_\eta \mid \eta < \kappa\}$. We have $S_0 = P_0^{1/\delta} \neq 0$. The function $S_\kappa(a) - \delta^{-1}P_0^{\alpha-1-\gamma\kappa/\delta-\gamma\beta/\delta}P_\kappa(a)$ is constant.*

Moreover, if the assumption of Lemma 5.6 is satisfied, we have the following.

- *If $S_\eta \neq 0$ and $\eta < \kappa$, we have $\text{ord}_{\mathfrak{p}_0}(\eta) > \text{ord}_{\mathfrak{p}_0}(\kappa)$.*
- *We have $S_\kappa(a) = \delta^{-1}P_0^{\alpha-1-\gamma\kappa/\delta-\gamma\beta/\delta}P_\kappa(a)$. In particular, we have $S_\kappa(0) \neq 0$.* ■

5.3.7 Proof of Proposition 5.3 and Proposition 5.4

For $1 \leq i \leq j \leq m$, we define $\mathbf{Y}_{i,j} = (\boldsymbol{\eta}_i, \omega_i, \dots, \boldsymbol{\eta}_j, \omega_j)$. We have $\mathbf{Y}_{1,\ell} = \mathbf{Y}_\ell$. We also set $\boldsymbol{\delta}(i, j) := \boldsymbol{\delta}(j)/\boldsymbol{\delta}(i-1)$. We have

$$\kappa(\mathbf{Y}_{i,j}) = \frac{\beta_i}{\boldsymbol{\delta}(i, i)} + \frac{\beta_{i+1}}{\boldsymbol{\delta}(i, i+1)} + \dots + \frac{\beta_j}{\boldsymbol{\delta}(i, j)}.$$

By the construction, we have $\kappa(\mathbf{Y}_{k,k}) = \beta_k \cdot \delta_k^{-1}$ and $\kappa(\mathbf{Y}_{1,\ell}) = \kappa(\mathbf{Y}_\ell)$. We also have the relation

$$\kappa(\mathbf{Y}_{i,j}) = \frac{\beta_i}{\delta_i} + \frac{\kappa(\mathbf{Y}_{i+1,j})}{\delta_i}.$$

We have the family of curves $F_{\mathbf{Y}_{i+1,j}}(a, y^{1/\boldsymbol{\delta}(i+1,j)}) : \mathcal{U} \times \Delta_\epsilon \longrightarrow (\mathbb{C}^2, x, y)$:

$$F_{\mathbf{Y}_{i+1,j}}(a, y^{1/\boldsymbol{\delta}(i+1,j)}) = (g_{\mathbf{Y}_{i+1,j}}(a, y^{1/\boldsymbol{\delta}(i+1,j)}), y).$$

We have the isomorphism $(U_{\mathbf{Y}_{i,i}}, u_{\mathbf{Y}_{i,i}}, v_{\mathbf{Y}_{i,i}}) \simeq (\mathbb{C}^2, x, y)$ given by the coordinate systems. We have the isomorphism $(U_{\mathbf{Y}_{i,i}}, u_{\mathbf{Y}_{i,i}}, v_{\mathbf{Y}_{i,i}}) \simeq (U_{\boldsymbol{\eta}_i-}, u_{\boldsymbol{\eta}_i-}, v_{\boldsymbol{\eta}_i-})$ given by $u_{\mathbf{Y}_{i,i}} = u_{\boldsymbol{\eta}_i-} - \omega_i$ and $v_{\mathbf{Y}_{i,i}} = v_{\boldsymbol{\eta}_i-}$. Hence, with respect to the coordinate system $(u_{\boldsymbol{\eta}_i-}, v_{\boldsymbol{\eta}_i-})$, $F_{\mathbf{Y}_{i+1,j}}$ is described as follows:

$$F_{\mathbf{Y}_{i+1,j}}(a, y^{1/\delta(i+1,j)}) = (\omega_i + g_{\mathbf{Y}_{i+1,j}}(a, y^{1/\delta(i+1,j)}), y).$$

We take the composite of $\psi_{\boldsymbol{\eta}_i-}$ and $F_{\mathbf{Y}_{i+1,j}}$. We clearly have $\delta(i+1, j)\delta_i = \delta(i, j)$. We take a $\delta(i, j)$ -root of $\omega_i^{\gamma_i}$. Then, we apply the construction in §5.3.6. We obtain the induced family

$$G_{\mathbf{Y}_{i,j}}(a, y^{1/\delta(i+1,j)\delta_i}) = (g'_{\mathbf{Y}_{i,j}}(a, y^{1/\delta(i+1,j)\delta_i}), y).$$

By the construction, it is equal to the induced family of curves $F_{\mathbf{Y}_{i,j}}$. Then, the claim of Proposition 5.3 follows from an induction and Lemma 5.8.

We assume that there exists a prime \mathfrak{p}_0 such that $\text{ord}_{\mathfrak{p}_0}(\kappa(\mathbf{Y}_m)) < 0$ and that $\text{ord}_{\mathfrak{p}_0}(\kappa(\mathbf{Y}_m)) < \text{ord}_{\mathfrak{p}_0}(\kappa(\mathbf{Y}_{m'}))$ for any $m' < m$ as in Proposition 5.4.

Lemma 5.9 *Let $m' < m$. We have $\text{ord}_{\mathfrak{p}_0}(\kappa(\mathbf{Y}_{m'+1,m})) < 0$ and $\text{ord}_{\mathfrak{p}_0}(\kappa(\mathbf{Y}_{m'+1,m})) < \text{ord}_{\mathfrak{p}_0}(\kappa(\mathbf{Y}_{m'+1,\ell}))$ for any $m' + 1 \leq \ell < m$.*

Proof Because $\kappa(\mathbf{Y}_{m'}) + \kappa(\mathbf{Y}_{m'+1,m}) \prod_{p \leq m'} \delta_p^{-1} = \kappa(\mathbf{Y}_m)$, we have

$$\text{ord}_{\mathfrak{p}_0} \left(\kappa(\mathbf{Y}_{m'+1,m}) \prod_{p \leq m'} \delta_p^{-1} \right) = \text{ord}_{\mathfrak{p}_0}(\kappa(\mathbf{Y}_m)). \quad (24)$$

Because $\kappa(\mathbf{Y}_{m'}) + \kappa(\mathbf{Y}_{m'+1,\ell}) \prod_{p \leq m'} \delta_p^{-1} = \kappa(\mathbf{Y}_\ell)$, we have

$$\text{ord}_{\mathfrak{p}_0} \left(\kappa(\mathbf{Y}_{m'+1,\ell}) \prod_{p \leq m'} \delta_p^{-1} \right) > \text{ord}_{\mathfrak{p}_0}(\kappa(\mathbf{Y}_m)).$$

Hence, we have

$$\text{ord}_{\mathfrak{p}_0} \left(\kappa(\mathbf{Y}_{m'+1,m}) \prod_{p \leq m'} \delta_p^{-1} \right) < \text{ord}_{\mathfrak{p}_0} \left(\kappa(\mathbf{Y}_{m'+1,\ell}) \prod_{p \leq m'} \delta_p^{-1} \right)$$

We obtain $\text{ord}_{\mathfrak{p}_0}(\kappa(\mathbf{Y}_{m'+1,m})) < \text{ord}_{\mathfrak{p}_0}(\kappa(\mathbf{Y}_{m'+1,\ell}))$.

Suppose that $\text{ord}_{\mathfrak{p}_0}(\kappa(\mathbf{Y}_{m'+1,m})) \geq 0$. Because of (24), we have

$$\text{ord}_{\mathfrak{p}_0} \left(\prod_{p \leq m'} \delta_p \right) \geq |\text{ord}_{\mathfrak{p}_0}(\kappa(\mathbf{Y}_m))|.$$

Put $\ell_0 := \min\{m' \mid \text{ord}_{\mathfrak{p}_0}(\prod_{p \leq m'} \delta_p) \geq |\text{ord}_{\mathfrak{p}_0}(\kappa(\mathbf{Y}_m))|\}$. Because $\text{g.c.d.}(\mathfrak{p}_0, \delta_{\ell_0}) = \mathfrak{p}_0$, we have $\text{g.c.d.}(\mathfrak{p}_0, \beta_{\ell_0}) = 1$.

Because $\kappa(\mathbf{Y}_{\ell_0}) = \kappa(\mathbf{Y}_{\ell_0-1}) + \prod_{i=1}^{\ell_0} \delta_i^{-1} \beta_{\ell_0}$, we have

$$\text{ord}_{\mathfrak{p}_0}(\kappa(\mathbf{Y}_{\ell_0})) = \text{ord}_{\mathfrak{p}_0} \left(\prod_{i=1}^{\ell_0} \delta_i^{-1} \beta_{\ell_0} \right) = -\text{ord}_{\mathfrak{p}_0}(\delta_1 \cdots \delta_{\ell_0}) \leq \text{ord}_{\mathfrak{p}_0}(\kappa(\mathbf{Y}_m))$$

It contradicts with the assumption $\text{ord}_{\mathfrak{p}_0} \kappa(\mathbf{Y}_{m'}) > \text{ord}_{\mathfrak{p}_0} \kappa(\mathbf{Y}_m)$ for $m' < m$. Hence, we can conclude $\text{ord}_{\mathfrak{p}_0}(\kappa(\mathbf{Y}_{m'+1,m})) < 0$. ■

Then, we obtain the claim of Proposition 5.4 by using an induction and Lemma 5.8. ■

6 Cross points in the surface case

6.1 Positively linear subsets

6.1.1 Subsets of $S^1 \times S^1$

For $(p, q) \in \mathbb{Z}_{>0}^2$ and $\phi = (\phi_1, \phi_2) \in S^1 \times S^1$, we set $H(p, q, \phi) := \{(\theta_1, \theta_2) \in S^1 \times S^1 \mid p(\theta_1 - \phi_1) = q(\theta_2 - \phi_2)\}$. Let A be a closed subanalytic subset in $S^1 \times S^1$ such that A is purely 1 dimensional, i.e., any open subset of A is one dimensional. We say that A is called positively linear if we have (p_i, q_i, ϕ_i) ($i = 1, \dots, N$) such that A is contained in $\bigcup_{i=1}^N H(p_i, q_i, \phi_i)$.

Lemma 6.1 *Let $B \subset S^1 \times S^1$ be a closed subanalytic subset such that B is purely 1 dimensional. Then, we have a decomposition $B = B_1 \cup B_2$ such that (i) B_1 is positively linear, (ii) $\dim(B_1 \cap B_2) = 0$, (iii) any open subset of B_2 is not positively linear.*

Proof Let B_1 be the union of positively linear subsets in B . It is enough to prove that B_1 is subanalytic. Let P be any point of B . Because $\dim B = 1$, B is semianalytic. (See [2].) Let U_P be a small neighbourhood of P in $(S^1)^2$. Let $I_\epsilon := [0, \epsilon]$. If U_P is sufficiently small, we have analytic maps $\kappa_i : I_\epsilon \rightarrow U_P$ ($i = 1, \dots, m$) such that $U_P \cap B = \bigcup_{i=1}^m \kappa_i(I_\epsilon)$. Then, the claim of the lemma is clear. \blacksquare

Let A_+ and A_- be the matrices as in (22). For any $\eta \in \{+, -\}^\ell$, we set $A_\eta := A_{\eta_1} A_{\eta_2} \cdots A_{\eta_\ell}$. We regard $S^1 \times S^1$ as $(\mathbb{R} \times \mathbb{R})/(\mathbb{Z} \times \mathbb{Z})$. We define the isomorphism $\Psi_\eta : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ by $\Psi_\eta \mathbf{v} = A_\eta \mathbf{v}$ for $\mathbf{v} = {}^t(v_1, v_2) \in \mathbb{R}^2$. The induced isomorphism $S^1 \times S^1 \rightarrow S^1 \times S^1$ is also denoted by Ψ_η .

Lemma 6.2 *Suppose that $pq \leq 0$. Then, we have $\Psi_\eta^{-1} H(p, q, \phi) \subset H(p', q', \phi')$ for some (p', q', ϕ') such that $p'q' \leq 0$.*

Proof It is enough to check the claim in the case $\eta = (\eta_1) \in \{+, -\}$, which can be checked directly. \blacksquare

Lemma 6.3 *For any positively linear subset $Z \subset S^1 \times S^1$, there exists ℓ_0 such that the following holds for any $\eta \in \{+, -\}^\ell$ if $\ell \geq \ell_0$:*

- $\Psi_\eta^{-1}(Z)$ does not contain positively linear subsets.

Proof It is enough to consider the case Z is an open subset of $H(p, q, \phi)$ for $(p, q) \in \mathbb{Z}_{>0}^2$ and $\phi = (\phi_1, \phi_2) \in S^1 \times S^1$. We use an induction on $p + q$. We may assume that $p \geq q$.

We consider the isomorphism $\Psi_{\eta_1}^{-1} : S^1 \times S^1 \rightarrow S^1 \times S^1$. If $\eta_1 = +$, $\Psi_{\eta_1}^{-1}(Z)$ is an open subset of $H(p - q, q, (0, \phi_2 - pq^{-1}\phi_1))$. Because $p - q + q < p + q$, we can apply the assumption of the induction to $\Psi_{\eta_1}^{-1}(Z)$.

Let us consider the case $\eta_1 = -$. Then, $\Psi_{\eta_1}^{-1}(Z)$ is an open subset of $H(p, q - p, (\phi_1 - qp^{-1}\phi_2, 0))$. Because $q - p$ is non-positive, we obtain the claim in this case by Lemma 6.2. \blacksquare

6.1.2 Subsets in the oriented real blowing up of complex surfaces

Let X be any complex surface with a normal crossing hypersurface H . Let $\varpi : \widetilde{X}(H) \rightarrow X$ be the oriented blowing up of X along H . Let P be any cross point of H . We take a holomorphic coordinate neighbourhood (X_P, z_1, z_2) around P such that $H = \bigcup_{i=1,2} \{z_i = 0\}$. Let $z_i = r_i \exp(\sqrt{-1}\theta_i)$ be the polar coordinate. It induces an isomorphism $\Phi_{\mathbf{z}} : \varpi^{-1}(P) \simeq S^1 \times S^1$. We say a locally closed purely 1-dimensional subanalytic subset Z in $\varpi^{-1}(P)$ is positively linear if $\Phi_{\mathbf{z}}(Z) \subset S^1 \times S^1$ is positively linear.

For another choice of holomorphic coordinate system (w_1, w_2) such that $\{w_i = 0\} = \{z_i = 0\}$, the isomorphism $\Phi_{\mathbf{w}} \circ \Phi_{\mathbf{z}}^{-1}$ is a translation $(\theta_1, \theta_2) \mapsto (\theta_1 + \theta_1^{(0)}, \theta_2 + \theta_2^{(0)})$. Hence, the condition of positively linear subsets are defined for purely 1-dimensional subanalytic subsets in $\varpi^{-1}(P)$, independently from the choice of the holomorphic coordinate system.

Let $\eta \in \{+, -\}^\ell$. We have the morphism $\psi_\eta : (\text{Bl}_\eta \mathbb{C}^2, P_\eta) \rightarrow (\mathbb{C}^2, O)$. Let $H := \{x = 0\} \cup \{y = 0\} \subset \mathbb{C}^2$. We set $H_\eta := \psi_\eta^{-1}(H) \subset \text{Bl}_\eta \mathbb{C}^2$. Let $\varpi : \widetilde{\mathbb{C}^2}(H) \rightarrow \mathbb{C}^2$ and $\varpi_\eta : \widetilde{\text{Bl}_\eta \mathbb{C}^2}(H_\eta) \rightarrow \text{Bl}_\eta \mathbb{C}^2$ be the oriented

real blowing up. We have the induced morphism $\widetilde{\psi}_\eta : \widetilde{\text{Bl}_\eta \mathbb{C}^2(H_\eta)} \longrightarrow \widetilde{\mathbb{C}^2(H)}$. We obtain the morphism $\Psi_\eta : \varpi_\eta^{-1}(P_\eta) \longrightarrow \varpi^{-1}(O)$ as the restriction of $\widetilde{\psi}_\eta$.

Lemma 6.4 *For any positively linear subset $Z \subset \varpi^{-1}(O)$, we have ℓ_0 such that $\Psi_\eta^{-1}(Z)$ does not contain a positively linear subset for any $\eta \in \{+, -\}^\ell$ if $\ell \geq \ell_0$.*

Proof It follows from the description of the morphism ψ_η in §5.2.2 and Lemma 6.3. ■

6.2 Separation of cross points from subanalytic hypersurface

We set $H = \{z = 0\} \cup \{w = 0\} \subset \mathbb{C}^2$. Let $\varpi : \widetilde{\mathbb{C}^2(H)} \longrightarrow \mathbb{C}^2$ be the oriented real blowing up. For any $\eta \in \{+, -\}^\ell$, we have the space $\text{Bl}_\eta \mathbb{C}^2$ with the base point P_η and the morphism $\psi_\eta : \text{Bl}_\eta \mathbb{C}^2 \longrightarrow \mathbb{C}^2$. Set $H_\eta := \psi_\eta^{-1}(H)$. Let $\varpi_\eta : \widetilde{\text{Bl}_\eta \mathbb{C}^2(H_\eta)} \longrightarrow \text{Bl}_\eta(\mathbb{C}^2)$ be the oriented real blowing up along H_η . We have the induced morphism $\widetilde{\psi}_\eta : \widetilde{\text{Bl}_\eta \mathbb{C}^2(H_\eta)} \longrightarrow \widetilde{\mathbb{C}^2(H)}$. The following proposition is a corollary of Proposition 6.6 in §6.2.1 below.

Proposition 6.5 *Let G be a closed subanalytic subset in $\widetilde{\mathbb{C}^2(H)}$ with $\dim_{\mathbb{R}} G \leq 3$. Then, there exists ℓ_0 such that the following holds for any $\eta \in \{+, -\}^\ell$ if $\ell \geq \ell_0$.*

- Let G_η denote the strict transform of G with respect to $\widetilde{\psi}_\eta$, i.e., G_η is the closure of $\psi_\eta^{-1}(G \setminus H)$ in $\widetilde{\text{Bl}_\eta \mathbb{C}^2(H_\eta)}$. Then, we have $\dim_{\mathbb{R}}(G_\eta \cap \varpi_\eta^{-1}(P_\eta)) \leq 1$.

6.2.1 Separation of cross points from subanalytic hypersurfaces

We set $\overline{\mathbb{R}}_{\geq 0} := \mathbb{R}_{\geq 0} \cup \{\infty\}$. Set $Y := \overline{\mathbb{R}}_{\geq 0}^2 \times (S^1)^2$. Let $W \subset Y$ be a closed subanalytic subset with $\dim_{\mathbb{R}} W = 3$.

Proposition 6.6 *We have a non-negative integer ℓ_0 and a closed subanalytic subset $Z_1 \subset (S^1)^2$ with $\dim_{\mathbb{R}} Z_1 \leq 1$ such that the following holds for any $\eta \in \{+, -\}^\ell$ if $\ell \geq \ell_0$:*

- Let W_η denote the strict transform of W with respect to $\psi_{\eta, \overline{\mathbb{R}}_{\geq 0}} \times \text{id} : \text{Bl}_\eta \overline{\mathbb{R}}_{\geq 0}^2 \times (S^1)^2 \longrightarrow \overline{\mathbb{R}}_{\geq 0}^2 \times (S^1)^2$, i.e., W_η denote the closure of $W \cap (\mathbb{R}_{>0}^2 \times (S^1)^2)$ in $\text{Bl}_\eta \overline{\mathbb{R}}_{\geq 0}^2 \times (S^1)^2$. Then, W_η does not intersect with $\{P_\eta\} \times ((S^1)^2 \setminus Z_1)$.

Proof Let $p : Y \longrightarrow \overline{\mathbb{R}}_{\geq 0}^2 \times (S^1)^2$ be given by $p(t_1, t_2, \theta_1, \theta_2) = (t_2, \theta_1, \theta_2)$.

Lemma 6.7 *We have a closed subanalytic subset $Z \subset \overline{\mathbb{R}}_{\geq 0}^2 \times (S^1)^2$ with $\dim Z \leq 2$ such that the following holds:*

- $p^{-1}(Z)$ contains the singular locus of W .
- For any $Q \in W \setminus p^{-1}(Z)$, the derivative of $p|_W$ at Q is an isomorphism.

Proof We have a proper morphism of real analytic manifolds $\varphi : \widetilde{W} \longrightarrow \overline{\mathbb{R}}_{\geq 0}^2 \times (S^1)^2$ such that (i) $\varphi(\widetilde{W}) = W$, (ii) $\dim_{\mathbb{R}} \widetilde{W} = 3$. Let $Z_0 \subset \widetilde{W}$ be the set of critical points of $p \circ \varphi$. We have $\dim(p \circ \varphi)(Z_0) \leq 2$. Let Z be the union of $p(\text{Sing}(W))$ and $(p \circ \varphi)(Z_0)$. Then, the claim of the lemma is clear. ■

By enlarging Z , we may assume that each connected component \mathcal{C} of $(\overline{\mathbb{R}}_{\geq 0}^2 \times (S^1)^2) \setminus Z$ is simply connected. The map $W \cap p^{-1}(\mathcal{C}) \longrightarrow \mathcal{C}$ is proper and a local diffeomorphism. We have the decomposition $W \cap p^{-1}(\mathcal{C}) = \coprod \mathcal{C}_i$ such that the induced maps $\mathcal{C}_i \longrightarrow \mathcal{C}$ are isomorphisms. We obtain the subanalytic functions $f_1^{\mathcal{C}}, \dots, f_m^{\mathcal{C}}$ on $(\mathcal{C}, \overline{\mathbb{R}}_{\geq 0} \times (S^1)^2)$ such that $W \cap p^{-1}(\mathcal{C})$ is the union of the graph $\Gamma(f_i^{\mathcal{C}})$ of $f_i^{\mathcal{C}}$.

We set $V_{\mathcal{C}} := \overline{\mathcal{C}} \cap (\{0\} \times (S^1)^2)$. Let $V_{\mathcal{C}}^\circ$ denote the interior part of $V_{\mathcal{C}} \subset (S^1)^2$. Let $\overline{V_{\mathcal{C}}^\circ}$ be the closure of $V_{\mathcal{C}}^\circ$ in $(S^1)^2$.

Lemma 6.8 *We have a closed subanalytic subset $Z_{1,\mathcal{C}} \subset \overline{V_{\mathcal{C}}^\circ}$ with $\dim_{\mathbb{R}} Z_{1,\mathcal{C}} \leq 1$ such that the following holds.*

- The functions f_j^c are ramified analytic around any point of $V_C^\circ \setminus Z_{1,C}$.
- For each $Q \in V_C^\circ \setminus Z_{1,C}$, we have rational numbers $b_j(Q)$ and positive rational numbers $c_j(Q)$ such that

$$f_j^c = t^{b_j(Q)} \cdot (g_{0,j}^Q(\theta_1, \theta_2) + t^{c_j(Q)} g_{1,j}^Q(\theta_1, \theta_2, t)).$$

Here, $g_{0,j}^Q$ are nowhere vanishing analytic functions on a neighbourhood of Q in V_C° , and $g_{1,j}^Q$ are bounded ramified analytic functions on a neighbourhood of Q in $\mathbb{R}_{\geq 0} \times (S^1)^2$. The numbers $b_j(Q)$ and $c_j(Q)$ are constants on the connected components of $V_C^\circ \setminus Z_1$.

Proof By Lemma 2.2, we have a closed subanalytic subset $Z'_1 \subset \overline{V_C^\circ}$ for which the first claim holds. Around Q , we have the expression

$$f_j^c = t^{b_j(Q)} \cdot (g_{0,j}(\theta_1, \theta_2) + t^{c_j(Q)} g_{1,j}(\theta_1, \theta_2, t)).$$

By enlarging Z'_1 , we may assume that $g_0(\theta_1, \theta_2)$ is nowhere vanishing, and the second claim follows. ■

We set $Z_1 := \bigcup_C Z_{1,C}$. It is a closed subanalytic subset in $\{0\} \times (S^1)^2 \subset \mathbb{R}_{\geq 0} \times (S^1)^2$. Let \mathcal{B} denote the set of rational numbers of the form $b_j(Q)$ for some $Q \in V_C$. It is a finite set.

For any $\boldsymbol{\eta} \in \{+, -\}^\ell$, we have the space $\text{Bl}_{\boldsymbol{\eta}} \mathbb{R}_{\geq 0}^2$ with the base point $P_{\boldsymbol{\eta}}$ and the morphism $\psi_{\boldsymbol{\eta}, \mathbb{R}_{\geq 0}^2} : \text{Bl}_{\boldsymbol{\eta}} \mathbb{R}_{\geq 0}^2 \rightarrow \mathbb{R}_{\geq 0}^2$. The following lemma is well known and easy to see.

Lemma 6.9 *Let b be any positive rational number. We have positive integers p, q such that $b = p/q$ and $\text{g.c.d.}(p, q) = 1$. For any positive numbers α and β , let $\rho_{\alpha, \beta}(t) : I_\epsilon \rightarrow \mathbb{R}_{\geq 0}^2$ be given by $\rho_{\alpha, \beta}(t) = (\alpha t^p, \beta t^q)$. There exists ℓ_1 such that the following holds for any $\boldsymbol{\eta} \in \{+, -\}^\ell$ if $\ell \geq \ell_1$.*

- The strict transform of $\rho_{\alpha, \beta}$ with respect to $\psi_{\boldsymbol{\eta}, \mathbb{R}_{\geq 0}^2}$ does not intersect with $P_{\boldsymbol{\eta}}$. ■

Let us finish the proof of Proposition 6.6. Take any $Q \in (\{0\} \times (S^1)^2) \setminus Z_1$. Locally around Q , W is described as the union of the closure $\overline{\Gamma(f_j^c)}$ of the graph $\Gamma(f_j^c)$. If $b_j(Q) \leq 0$, then $\overline{\Gamma(f_j^c)}$ does not intersect with $(0, 0) \times (S^1)^2$. By applying Lemma 6.9 to each $b \in \mathcal{B} \cap \mathbb{Q}_{>0}$, we obtain the claim of Proposition 6.6. ■

Note that $\mathbb{R}_{\geq 0}^2 \times (S^1)^2$ is a compactification of $\widetilde{\mathbb{C}^2}(H)$. Proposition 6.5 follows from Proposition 6.6 and the expression of $\psi_{\boldsymbol{\eta}}$ in §5.2.2. ■

6.3 Lifting with respect to sequences of local real blowings up

6.3.1 Some statements

Let M be any 2-dimensional real analytic manifold. Let $\phi : W \rightarrow \mathbb{R}^2 \times M$ be the composition of local real blowings up i.e., we have a factorization of ϕ as follows:

$$W = W_k \xrightarrow{\phi_k} W_{k-1} \xrightarrow{\phi_{k-1}} \dots \xrightarrow{\phi_2} W_1 \xrightarrow{\phi_1} W_0 = \mathbb{R}^2 \times M.$$

Moreover, we have subanalytic open subsets $U_i \subset W_i$ and closed real analytic submanifolds $C_i \subset U_i$ such that $W_{i+1} \rightarrow W_i$ are the real blowing up of U_i along C_i . For simplicity, we assume that C_i is subanalytic in W_i . Let $q_i : W_i \rightarrow M$ denote the morphism obtained as the composite of the induced morphism $W_i \rightarrow W_0$ and the projection $W_0 \rightarrow M$.

We take a number $\ell > k$ and an element $\boldsymbol{\eta} = (\eta_1, \dots, \eta_\ell) \in \{+, -\}^\ell$. For each $m \leq \ell$, we set $\boldsymbol{\eta}_m := (\eta_1, \dots, \eta_m)$. We shall prove the following proposition in §6.3.2.

Proposition 6.10 *We have a closed subanalytic subset $Z \subset M$ with $\dim Z = 1$ with the following property.*

- Each connected component of $M \setminus Z$ is simply connected.
- The restriction of q_i to $C_i \setminus q_i^{-1}(Z)$ is proper and a local diffeomorphism to $M \setminus Z$.

- Let \mathcal{D} be any connected component of $M \setminus Z$. For each $i \leq k$, we have a non-negative number $\ell(\mathcal{D}, i) \leq \ell$, closed subanalytic subset $L_{\mathcal{D}, i} \subset q_i^{-1}(\mathcal{D})$ with $\dim L_{\mathcal{D}, i} \leq 3$, and an open immersion $\iota_{\mathcal{D}, i} : q_i^{-1}(\mathcal{D}) \setminus L_{\mathcal{D}, i} \rightarrow \mathrm{Bl}_{\mathbf{n}_{\ell(\mathcal{D}, i)}} \mathbb{R}^2 \times \mathcal{D}$.
- We have $L_{\mathcal{D}, 0} = \emptyset$, and $L_{\mathcal{D}, i} \subset \phi_i^{-1}(L_{\mathcal{D}, i-1})$.
- The following diagram is commutative:

$$\begin{array}{ccc} q_{i+1}^{-1}(\mathcal{D}) \setminus L_{\mathcal{D}, i+1} & \xrightarrow{\iota_{\mathcal{D}, i+1}} & \mathrm{Bl}_{\mathbf{n}_{\ell(\mathcal{D}, i+1)}} \mathbb{R}^2 \times \mathcal{D} \\ \downarrow \phi_{i+1} & & \downarrow \\ q_i^{-1}(\mathcal{D}) \setminus L_{\mathcal{D}, i} & \xrightarrow{\iota_{\mathcal{D}, i}} & \mathrm{Bl}_{\mathbf{n}_{\ell(\mathcal{D}, i)}} \mathbb{R}^2 \times \mathcal{D} \end{array}$$

- Set $U_{i|\mathcal{D}} := U_i \cap q_i^{-1}(\mathcal{D})$. Then, we have either one of the following:
 - $\iota_{\mathcal{D}, i}(U_{i|\mathcal{D}} \setminus L_{\mathcal{D}, i}) \supset P_{\mathbf{n}_{\ell(\mathcal{D}, i)}} \times \mathcal{D}$.
 - $\iota_{\mathcal{D}, i}(U_{i|\mathcal{D}} \setminus L_{\mathcal{D}, i}) \cap (P_{\mathbf{n}_{\ell(\mathcal{D}, i)}} \times \mathcal{D}) = \emptyset$.
- If $\iota_{\mathcal{D}, i}(U_{i|\mathcal{D}} \setminus L_{\mathcal{D}, i}) \supset P_{\mathbf{n}_{\ell(\mathcal{D}, i)}} \times \mathcal{D}$, then we have $\iota_{\mathcal{D}, i+1}(q_{i+1}^{-1}(\mathcal{D}) \setminus L_{\mathcal{D}, i+1}) \supset P_{\mathbf{n}_{\ell(\mathcal{D}, i+1)}} \times \mathcal{D}$.

Before giving a proof of Proposition 6.10, we give a consequence. Let $\mathrm{Crit}(\phi) \subset \mathbb{R}^2 \times M$ be the set of the critical values of ϕ . We have $\dim_{\mathbb{R}} \mathrm{Crit}(\phi) \leq 3$.

Proposition 6.11 *Let Q be a point of $M \setminus Z$. Suppose that we have an analytic path $\gamma : [0, 1] \rightarrow \mathrm{Bl}_{\mathbf{n}} \mathbb{R}^2 \times M$ such that the following holds:*

- We have $\gamma(0) = (P_{\mathbf{n}}, Q)$, and $\psi_{\mathbf{n}} \circ \gamma(t) \in (\mathbb{R}^2 \times M) \setminus \mathrm{Crit}(\phi)$ for $t > 0$.
- We have a lift $\gamma' : [0, 1] \rightarrow W$ such that $\phi \circ \gamma' = \psi_{\mathbf{n}} \circ \gamma$.

Then, we have the following:

- a small neighbourhood \mathcal{U} of $(P_{\mathbf{n}}, Q)$ in $\mathrm{Bl}_{\mathbf{n}} \mathbb{R}^2 \times M$,
- a small neighbourhood $W_{\gamma'(0)}$ of $\gamma'(0)$ in W ,
- a non-negative integer $i(Q) \leq \ell$,
- an open embedding $\iota_Q : W_{\gamma'(0)} \rightarrow \mathrm{Bl}_{\mathbf{n}_{i(Q)}} \mathbb{R}^2 \times M$, such that (i) $\phi = (\psi_{\mathbf{n}_{i(Q)}} \times \mathrm{id}) \circ \iota_Q$, (ii) the morphism $\mathrm{Bl}_{\mathbf{n}} \mathbb{R}^2 \times M \rightarrow \mathrm{Bl}_{\mathbf{n}_{i(Q)}} \mathbb{R}^2 \times M$ induces the real analytic map $\mathcal{U} \rightarrow \iota_Q(W_{\gamma'(0)})$. We also have $\phi|_{W_{\gamma'(0)}} = \psi_{\mathbf{n}_{i(Q)}} \circ \iota_Q$.

Proof We have the connected component \mathcal{D} of $M \setminus Z$ such that $((0, 0), Q) \in U_{0|\mathcal{D}}$. Let $\gamma_0 := \psi_{\mathbf{n}} \circ \gamma$. We have $P_{\mathbf{n}_{\ell(\mathcal{D}, 1)}} \times \mathcal{D} \subset \iota_{\mathcal{D}, 1}(q_1^{-1}(\mathcal{D}) \setminus L_{\mathcal{D}, 1})$. We have a lift γ_1 of γ_0 to W_1 . It implies that $(P_{\mathbf{n}_{\ell(\mathcal{D}, 1)}} \times \mathcal{D}) \cap \iota_{\mathcal{D}, 1}(U_{1|\mathcal{D}} \setminus L_{\mathcal{D}, 1}) \neq \emptyset$. Hence, we have $(P_{\mathbf{n}_{\ell(\mathcal{D}, 1)}} \times \mathcal{D}) \subset \iota_{\mathcal{D}, 1}(U_{1|\mathcal{D}} \setminus L_{\mathcal{D}, 1})$. Inductively, we obtain that $P_{\mathbf{n}_{\ell(\mathcal{D}, i)}} \times \mathcal{D} \subset \iota_{\mathcal{D}, i}(U_{i|\mathcal{D}} \setminus L_{\mathcal{D}, i})$. In particular, $\iota_{\mathcal{D}, k}(q_k^{-1}(\mathcal{D}) \setminus L_{\mathcal{D}, k})$ is an open neighbourhood of $P_{\mathbf{n}_{\ell(\mathcal{D}, k)}} \times \mathcal{D}$. Then, the claim of Proposition 6.11 follows. \blacksquare

6.3.2 Proof of Proposition 6.10

First, we can take a closed subanalytic subset $Z \subset M$ with $\dim_{\mathbb{R}} Z \leq 1$ satisfying the following conditions.

- $M \setminus Z$ is simply connected.
- The restriction of q_i to $C_i \setminus q_i^{-1}(Z)$ are proper and local diffeomorphisms to $M \setminus Z$ for any i .

In the following, we shall enlarge Z . We note that $\dim C_i \leq 2$.

Let \mathcal{D} be a connected component of $M \setminus Z$. Let $C_{0|\mathcal{D}} := C_0 \cap q_0^{-1}(\mathcal{D})$. We have the decomposition into connected components $C_{0|\mathcal{D}} = \coprod_{j \in \Lambda(0, \mathcal{D})} C_{0, \mathcal{D}, j}$. After enlarging Z , we may assume that either one of the following holds for each $j \in \Lambda(0, \mathcal{D})$; (i) $C_{0, \mathcal{D}, j} \cap ((0, 0) \times \mathcal{D}) = \emptyset$, (ii) $C_{0, \mathcal{D}, j} = (0, 0) \times \mathcal{D}$. We may also assume that either one of the following holds; (i) $U_0 \cap ((0, 0) \times \mathcal{D}) = \emptyset$, (ii) $U_0 \supset (0, 0) \times \mathcal{D}$. We set $\Lambda^\circ(0, \mathcal{D}) := \{j \mid C_{0, \mathcal{D}, j} = (0, 0) \times \mathcal{D}\}$ and $\Lambda^\perp(0, \mathcal{D}) := \Lambda(0, \mathcal{D}) \setminus \Lambda^\circ(0, \mathcal{D})$. We set $C_{0, \mathcal{D}}^\perp := \coprod_{j \in \Lambda^\perp(0, \mathcal{D})} C_{0, \mathcal{D}, j}$.

We set $\ell(1, \mathcal{D}) := 1$ if $\Lambda^\circ(0, \mathcal{D}) \neq \emptyset$, or $\ell(1, \mathcal{D}) := 0$ if $\Lambda^\circ(0, \mathcal{D}) = \emptyset$. We set $L_{1, \mathcal{D}} := \phi_1^{-1}(C_{0, \mathcal{D}}^\perp) \subset q_1^{-1}(\mathcal{D})$. Then, by the construction, we have the natural open immersion $\iota_{1, \mathcal{D}} : q_1^{-1}(\mathcal{D}) \setminus L_{1, \mathcal{D}} \rightarrow \text{Bl}_{\mathbf{n}_{\ell(1, \mathcal{D})}} \times \mathcal{D}$. If $(0, 0) \times \mathcal{D} \subset U_{0|\mathcal{D}} \times \mathcal{D}$, then we have $P_{\mathbf{n}_{\ell(1, \mathcal{D})}} \times \mathcal{D} \subset \iota_{1, \mathcal{D}}(q_1^{-1}(\mathcal{D}) \setminus L_{1, \mathcal{D}})$.

We have $C_1 \subset W_1$. We have the decomposition into connected components $C_{1|\mathcal{D}} = \coprod_{j \in \Lambda(1, \mathcal{D})} C_{1, \mathcal{D}, j}$. After enlarging Z , we may assume that either one of the following for each $j \in \Lambda(1, \mathcal{D})$; (i) $C_{1, \mathcal{D}, j} \cap L_{1, \mathcal{D}} = \emptyset$, (ii) $C_{1, \mathcal{D}, j} \subset L_{1, \mathcal{D}}$. We may assume that either one of the following for each $j \in \Lambda(1, \mathcal{D})$; (i) $\iota_{1, \mathcal{D}}(C_{1, \mathcal{D}, j} \setminus L_{1, \mathcal{D}}) \cap (P_{\mathbf{n}_{\ell(1, \mathcal{D})}} \times \mathcal{D}) = \emptyset$, (ii) $\iota_{1, \mathcal{D}}(C_{1, \mathcal{D}, j} \setminus L_{1, \mathcal{D}}) = P_{\mathbf{n}_{\ell(1, \mathcal{D})}} \times \mathcal{D}$. We may also assume that either one of the following holds: (i) $\iota_{1, \mathcal{D}}(U_{1|\mathcal{D}}) \supset (P_{\mathbf{n}_{\ell(1, \mathcal{D})}} \times \mathcal{D})$, (ii) $\iota_{1, \mathcal{D}}(U_{1|\mathcal{D}}) \cap (P_{\mathbf{n}_{\ell(1, \mathcal{D})}} \times \mathcal{D}) = \emptyset$.

We set $\Lambda^\circ(1, \mathcal{D}) := \{j \in \Lambda(1, \mathcal{D}) \mid C_{1, \mathcal{D}, j} \cap L_{1, \mathcal{D}} \neq \emptyset, \iota_{1, \mathcal{D}}(C_{1, \mathcal{D}, j}) = P_{\mathbf{n}_{\ell(1, \mathcal{D})}} \times \mathcal{D}\}$ and $\Lambda^\perp(1, \mathcal{D}) := \Lambda(1, \mathcal{D}) \setminus \Lambda^\circ(1, \mathcal{D})$. We set $C_{1, \mathcal{D}}^\perp := L_{1, \mathcal{D}} \cup \coprod_{j \in \Lambda^\perp(1, \mathcal{D})} C_{1, \mathcal{D}, j}$. We set

$$\ell(2, \mathcal{D}) := \begin{cases} \ell(1, \mathcal{D}) + 1, & (\Lambda^\circ(1, \mathcal{D}) \neq \emptyset), \\ \ell(1, \mathcal{D}), & (\Lambda^\circ(1, \mathcal{D}) = \emptyset). \end{cases}$$

We set $L_{2, \mathcal{D}} := \phi_2^{-1}(C_{1, \mathcal{D}}^\perp)$. By the construction, we have the naturally defined open immersion $q_2^{-1}(\mathcal{D}) \setminus L_{2, \mathcal{D}} \rightarrow \text{Bl}_{\mathbf{n}_{\ell(2, \mathcal{D})}} \mathbb{R}_{\geq 0}^2 \times \mathcal{D}$. If $P_{\mathbf{n}_{\ell(1, \mathcal{D})}} \times \mathcal{D} \subset U_{1|\mathcal{D}} \setminus L_{1, \mathcal{D}}$, then we have $P_{\mathbf{n}_{\ell(2, \mathcal{D})}} \times \mathcal{D} \subset q_2^{-1}(\mathcal{D}) \setminus L_{2, \mathcal{D}}$ by construction.

Inductively, we construct the desired data. Suppose that we have already obtained $\ell(\mathcal{D}, i)$, $L_{\mathcal{D}, i} \subset q_i^{-1}(\mathcal{D})$, and an open immersion $\iota_{\mathcal{D}, i} : q_i^{-1}(\mathcal{D}) \setminus L_{\mathcal{D}, i} \rightarrow \text{Bl}_{\mathbf{n}_{\ell(\mathcal{D}, i)}} \mathbb{R}^2 \times \mathcal{D}$. We have $C_i \subset W_i$. We have the decomposition $C_{i|\mathcal{D}} = \coprod_{j \in \Lambda(i, \mathcal{D})} C_{i, \mathcal{D}, j}$. After enlarging Z , we may assume that either one of the following for each $j \in \Lambda(i, \mathcal{D})$; (i) $C_{i, \mathcal{D}, j} \cap L_{i, \mathcal{D}} = \emptyset$, or (ii) $C_{i, \mathcal{D}, j} \subset L_{i, \mathcal{D}}$. We may assume that either one of the following for each $j \in \Lambda(i, \mathcal{D})$; (i) $\iota_{i, \mathcal{D}}(C_{i, \mathcal{D}, j} \setminus L_{i, \mathcal{D}}) \cap (P_{\mathbf{n}_{\ell(i, \mathcal{D})}} \times \mathcal{D}) = \emptyset$, (ii) $\iota_{i, \mathcal{D}}(C_{i, \mathcal{D}, j} \setminus L_{i, \mathcal{D}}) = P_{\mathbf{n}_{\ell(i, \mathcal{D})}} \times \mathcal{D}$. We may also assume that either one of the following holds: (i) $\iota_{i, \mathcal{D}}(U_{i|\mathcal{D}}) \supset P_{\mathbf{n}_{\ell(i, \mathcal{D})}} \times \mathcal{D}$, (ii) $\iota_{i, \mathcal{D}}(U_{i|\mathcal{D}}) \cap (P_{\mathbf{n}_{\ell(i, \mathcal{D})}} \times \mathcal{D}) = \emptyset$.

We set $\Lambda^\circ(i, \mathcal{D}) := \{j \in \Lambda(i, \mathcal{D}) \mid \iota_{i, \mathcal{D}}(C_{i, \mathcal{D}, j}) = P_{\mathbf{n}_{\ell(i, \mathcal{D})}} \times \mathcal{D}\}$ and $\Lambda^\perp(i, \mathcal{D}) := \Lambda(i, \mathcal{D}) \setminus \Lambda^\circ(i, \mathcal{D})$. We set $C_{i, \mathcal{D}}^\perp := L_{i, \mathcal{D}} \cup \coprod_{j \in \Lambda^\perp(i, \mathcal{D})} C_{i, \mathcal{D}, j}$. We set

$$\ell(i+1, \mathcal{D}) := \begin{cases} \ell(i, \mathcal{D}) + 1, & (\Lambda^\circ(i, \mathcal{D}) \neq \emptyset), \\ \ell(i, \mathcal{D}), & (\Lambda^\circ(i, \mathcal{D}) = \emptyset). \end{cases}$$

We set $L_{i+1, \mathcal{D}} := \phi_{i+1}^{-1}(C_{i, \mathcal{D}}^\perp)$. By the construction, we have the naturally defined open immersion $q_{i+1}^{-1}(\mathcal{D}) \setminus L_{i+1, \mathcal{D}} \rightarrow \text{Bl}_{\mathbf{n}_{\ell(i+1, \mathcal{D})}} \mathbb{R}_{\geq 0}^2 \times \mathcal{D}$. If $P_{\mathbf{n}_{\ell(i, \mathcal{D})}} \times \mathcal{D} \subset U_{i|\mathcal{D}} \setminus L_{i, \mathcal{D}}$, then we have $P_{\mathbf{n}_{\ell(i+1, \mathcal{D})}} \times \mathcal{D} \subset q_{i+1}^{-1}(\mathcal{D}) \setminus L_{i+1, \mathcal{D}}$ by construction. In this way, the inductive construction can proceed. \blacksquare

6.4 Resolutions to ramified analytic functions

6.4.1 Ramified analytic functions on cornered real analytic manifolds

Let U be an open subset in $\mathbb{R}_{\geq 0}^m \times \mathbb{R}^\ell = \{(x_1, \dots, x_m, y_1, \dots, y_\ell)\}$. Set $D := U \cap \left(\bigcup_{i=1}^m \{x_i = 0\}\right)$. An analytic function f on $U \setminus D$ is called ramified analytic on U if f is expressed as follows around any point $Q \in D$:

$$f = \sum_{j_i \geq -N} a_{j_1, \dots, j_m}(y_1, \dots, y_\ell) \prod_{i=1}^m x_i^{j_i/\rho}.$$

Here, ρ is a positive integer, and $a_{j_1, \dots, j_m}(y_1, \dots, y_\ell)$ are analytic functions.

6.4.2 Statement

Set $H := \{x = 0\} \cup \{y = 0\}$ in \mathbb{C}^2 . Let $\varpi : \widetilde{\mathbb{C}^2}(H) \rightarrow \mathbb{C}^2$ denote the oriented real blowing up. Let W be a closed subanalytic subset in $\widetilde{\mathbb{C}^2}(H)$ with $\dim_{\mathbb{R}} W \leq 3$ such that the following holds.

- Let $W^\circ := W \cap (\mathbb{C}^2 \setminus H)$. Then, $\dim(\varpi^{-1}(O) \cap \overline{W^\circ}) \leq 1$.

Let f_i ($i = 1, \dots, n$) be continuous subanalytic functions on $(\widetilde{\mathbb{C}^2}(H) \setminus (W \cup \varpi^{-1}(H)), \widetilde{\mathbb{C}^2}(H))$.

For any $\eta \in \{+, -\}^\ell$, we have the space $\text{Bl}_\eta \mathbb{C}^2$ with the base point P_η and the morphism $\psi_\eta : \text{Bl}_\eta \mathbb{C}^2 \rightarrow \mathbb{C}^2$. We set $H_\eta := \psi_\eta^{-1}(H)$. Let $\varpi_\eta : \widetilde{\text{Bl}_\eta \mathbb{C}^2}(H_\eta) \rightarrow \text{Bl}_\eta \mathbb{C}^2$ be the oriented real blowing up along H_η . We have the morphism $\widetilde{\psi}_\eta : \widetilde{\text{Bl}_\eta \mathbb{C}^2}(H_\eta) \rightarrow \widetilde{\mathbb{C}^2}(H)$. As the restriction, we have the isomorphism $\Psi_\eta : \varpi_\eta^{-1}(P_\eta) \rightarrow \varpi^{-1}(O)$.

Proposition 6.12 *We have a positive number ℓ_0 and a closed subanalytic subset $Z \subset \varpi^{-1}(O)$ with $\dim_{\mathbb{R}} Z \leq 1$ such that the following holds for any $\eta \in \{+, -\}^\ell$ if $\ell \geq \ell_0$.*

- Z contains $\overline{W^\circ} \cap \varpi^{-1}(O)$.
- For any $Q \in \varpi_\eta^{-1}(P_\eta) \setminus \Psi_\eta^{-1}(Z)$, the functions $\widetilde{\psi}_\eta^*(f_i)$ are ramified analytic around Q .

Proof We naturally regard $\widetilde{\mathbb{C}^2}(H) = (\mathbb{R}_{\geq 0} \times S^1)^2$ which is naturally a closed subanalytic subset of the real analytic manifold $(\mathbb{R} \times S^1)^2$. We have the rectilinearization (W_α, ϕ_α) ($\alpha \in \Lambda$) for the subanalytic functions $\{f_i\}$. We have compact subsets $K_\alpha \subset W_\alpha$ such that $\bigcup_\alpha \phi_\alpha(K_\alpha)$ contains a neighbourhood of $\widetilde{\mathbb{C}^2}(H)$ in $(\mathbb{R} \times S^1)^2$. Each $\phi_\alpha : W_\alpha \rightarrow (\mathbb{R} \times S^1)^2$ is factorized as follows:

$$W_\alpha = W_{\alpha, k(\alpha)} \xrightarrow{\phi_\alpha^{(k(\alpha))}} W_{\alpha, k(\alpha)-1} \xrightarrow{\phi_\alpha^{(k(\alpha)-1)}} \cdots \rightarrow W_{\alpha, 2} \xrightarrow{\phi_\alpha^{(2)}} W_{\alpha, 1} \xrightarrow{\phi_\alpha^{(1)}} W_{\alpha, 0} = (\mathbb{R} \times S^1)^2 = \mathbb{R}^2 \times \varpi^{-1}(O)$$

We take $Z_\alpha^{(1)} \subset \varpi^{-1}(O)$ as in Proposition 6.10. Let $Z_\alpha^{(2)} = \bigcup \phi_\alpha(Q)$, where Q runs through the quadrants of W_α such that $\dim \phi_\alpha(Q) \leq 1$. We set $Z := \overline{W^\circ} \cup \bigcup_{\alpha \in \Lambda} (Z_\alpha^{(1)} \cup Z_\alpha^{(2)})$. Let $\text{Crit}(\phi_\alpha) \subset (\mathbb{R} \times S^1)^2$ denote the set of the critical values of ϕ_α .

We take ℓ_0 such that $\ell_0 > k(\alpha)$ for any α such that $\phi_\alpha(K_\alpha) \cap \varpi^{-1}(O) \neq \emptyset$. Take any $\ell \geq \ell_0$ and $\eta \in \{+, -\}^\ell$. Take any $Q \in \varpi_\eta^{-1}(P_\eta) \setminus \Psi_\eta^{-1}(Z)$. We can take a path $\gamma : [0, 1] \rightarrow \text{Bl}_\eta \mathbb{C}^2$ such that we have $\gamma(0) = Q$ and the following for any $t > 0$:

$$\widetilde{\psi}_\eta \circ \gamma(t) \in (\mathbb{R}_{>0} \times S^1)^2 \setminus \bigcup_{\alpha \in \Lambda} \text{Crit}(\phi_\alpha) = (\mathbb{C}^2 \setminus H) \setminus \bigcup_{\alpha \in \Lambda} \text{Crit}(\phi_\alpha).$$

Because $\bigcup_{\alpha \in \Lambda} \phi_\alpha(K_\alpha)$ contains a neighbourhood of $\widetilde{\mathbb{C}^2}(H)$, we can find α_0 and $\epsilon > 0$ such that $\widetilde{\psi}_\eta \circ \gamma([0, \epsilon]) \subset \phi_{\alpha_0}(K_{\alpha_0})$. We can find $\gamma' : [0, 1] \rightarrow W_{\alpha_0}$ such that $\phi_{\alpha_0} \circ \gamma'(t) = \widetilde{\psi}_\eta \circ \gamma(t)$ for any $t \in [0, \epsilon]$. We can take a small neighbourhood $W_{\alpha_0, \gamma'(0)}$ of $\gamma'(0)$ in W_{α_0} , an integer $i_0 \leq \ell_0$, and an embedding $\iota_0 : W_{\alpha_0, \gamma'(0)} \rightarrow \text{Bl}_{\eta_{i(Q)}} \mathbb{R}^2 \times (S^1)^2$ as in Proposition 6.11. In particular, we have $\phi_{\alpha_0|W_{\alpha_0, \gamma'(0)}} = (\psi_{\eta_{i(Q)}} \times \text{id}) \circ \iota_0$. Note that $\mathcal{V} := \phi_{\alpha_0}^{-1}(\widetilde{\mathbb{C}^2}(H) \setminus (W \cup \varpi^{-1}(H)))$ is rectilinearized with respect to the coordinate of W_{α_0} . Moreover, $\gamma'(0)$ is not contained in a quadrant whose dimension is strictly smaller than 2. Hence, $\iota_Q(W_{\alpha_0, \gamma'(0)} \cap \mathcal{V})$ is equal to $\iota_Q(W_{\alpha_0, \gamma'(0)}) \cap (\text{Bl}_{\eta_{i(Q)}} \mathbb{R}_{\geq 0}^2 \times (S^1)^2)$, and the pull back $(\iota_Q^{-1})^* \phi_{\alpha_0}^*(f_i)$ is expressed as the power series of $(u_{\eta_{i(Q)}, \mathbb{R}}^{1/\rho}, v_{\eta_{i(Q)}, \mathbb{R}}^{1/\rho}, \theta_1, \theta_2)$ for a positive integer ρ . Here, (θ_1, θ_2) is a natural coordinate of $(S^1)^2$, and $(u_{\eta_{i(Q)}, \mathbb{R}}^{1/\rho}, v_{\eta_{i(Q)}, \mathbb{R}}^{1/\rho})$ is as in §5.2.2.

If we take a small neighbourhood \mathcal{U}_Q of Q in $\widetilde{\text{Bl}_\eta \mathbb{C}^2}(H_\eta)$, we have the factorization of $\widetilde{\psi}_\eta|_{\mathcal{U}_Q}$ as follows:

$$\mathcal{U}_Q \xrightarrow{\Phi} \iota(W_{\alpha_0, \gamma'(0)} \cap \mathcal{V}) \xrightarrow{\phi_{\alpha_0} \circ \iota_Q^{-1}} \mathbb{R}^2 \times (S^1)^2.$$

Here, Φ is induced by $\psi_{\eta_{i(Q)}, \eta} : \text{Bl}_\eta \mathbb{R}^2 \rightarrow \text{Bl}_{\eta_{i(Q)}} \mathbb{R}^2$ and Ψ_η . It is easy to see that the pull back of $(\iota_Q^{-1})^* \phi_{\alpha_0}^*(f_i)$ via Φ is ramified analytic. Thus, we obtain the claim of Proposition 6.12. \blacksquare

6.5 Resolution for auxiliary conditions

6.5.1 Preliminary

Let X be a complex surface with a normal crossing hypersurface H . Let $\varpi : \tilde{X}(H) \rightarrow X$ denote the oriented real blowing up of X along H . Let $K \in \mathbb{E}_{\odot}^b(IC_{\mathbf{X}(H)})$. Let P be a cross point of H . We say that K satisfies the condition **(B)** around P if we have a filtration $\tilde{X}(H) = \tilde{X}(H)^{(0)} \supset \tilde{X}(H)^{(1)} \supset \dots$ for K such that the following holds.

(B) Let W denote the closure of $\tilde{X}(H)^{(1)} \setminus \partial\tilde{X}(H)$. Then, we have $\dim_{\mathbb{R}}(W \cap \varpi^{-1}(P)) \leq 1$. Moreover, $W \cap \varpi^{-1}(P)$ does not contain any positively linear subspace.

6.5.2 Statement

Let $X := \Delta^2$ and $H := \{x_1 = 0\} \cup \{x_2 = 0\}$, or $H := \{x_1 = 0\}$. For any $\eta \in \{+, -\}^\ell$, let $\text{Bl}_{\eta} X$ (resp. H_{η}) the inverse image of X (resp. H) by the morphism $\psi_{\eta} : \text{Bl}_{\eta} \mathbb{C}^2 \rightarrow \mathbb{C}^2$. The induced morphism $\text{Bl}_{\eta} X \rightarrow X$ is also denoted by ψ_{η} . We shall prove the following proposition.

Proposition 6.13 *For any $K \in \mathbb{E}_{\odot}^b(IC_{\mathbf{X}(H)})$, there exists a positive integer ℓ_0 such that the following holds for any $\eta \in \{+, -\}^\ell$ if $\ell \geq \ell_0$:*

- $\mathbb{E}\psi_{\eta}^{-1}K$ satisfies **(A)** and **(B)** around P_{η} .

(See §3.7 for the condition **(A)**.)

Proof Let us consider the following conditions.

(C1) Let $\varpi : \tilde{X}(H) \rightarrow X$ be the oriented real blowing up. We have a closed subanalytic subset $Z \subset \varpi^{-1}(O)$ with $\dim Z \leq 1$ such that for any $Q \in \varpi^{-1}(O) \setminus Z$ there exist a neighbourhood \mathcal{U}_Q in $\tilde{X}(H)$ and ramified analytic functions g_1^Q, \dots, g_m^Q on \mathcal{U}_Q and that the growth order of $\pi^{-1}(\mathbb{C}_{\mathcal{U}_Q}) \otimes K$ is controlled by g_1^Q, \dots, g_m^Q .

(C2) Moreover, Z does not contain any positively linear subset.

Lemma 6.14 *Suppose that the conditions **(C1)** and **(C2)** are satisfied. Then, we have a neighbourhood X' of O in X and a set of ramified irregular values \mathcal{I} at O for which the following holds.*

- Let $\varphi : \Delta \rightarrow X'$ be any holomorphic map. Then, we have $\text{Irr}(\mathbb{E}\varphi^{-1}K) = \varphi^*\mathcal{I}$.

Proof We use the polar coordinate $z_i = r_i e^{\sqrt{-1}\theta_i}$. We take a general point Q . We have the functions g_1^Q, \dots, g_m^Q as above. We have the expressions

$$g_i^Q = \sum_{j \geq -N_1} \sum_{k \geq -N_2} \alpha_{i,j,k}(\theta_1, \theta_2) r_1^{j/\rho} r_2^{k/\rho}.$$

We set

$$g_{i1}^Q = \sum_{-N_1 \leq j} \sum_{-N_1 \leq k < 0} \alpha_{i,j,k}(\theta_1, \theta_2) r_1^{j/\rho} r_2^{k/\rho}, \quad g_{i2}^Q = \sum_{-N_1 \leq j < 0} \sum_{-N_1 \leq k} \alpha_{i,j,k}(\theta_1, \theta_2) r_1^{j/\rho} r_2^{k/\rho}.$$

By taking a ramified covering, we may assume $\rho = 1$. By the argument in the proof of Lemma 4.1, we have meromorphic functions

$$f_{i1}^Q = \sum_{-N_1 \leq j} \sum_{-N_1 \leq k < 0} a_{i,j,k}^1 z_1^{j/\rho} z_2^{k/\rho}, \quad f_{i2}^Q = \sum_{-N_1 \leq j < 0} \sum_{-N_1 \leq k} a_{i,j,k}^2 z_1^{j/\rho} z_2^{k/\rho},$$

such that $\text{Re}(f_{ia}^Q) = g_{ia}^Q$. We have

$$\text{Re}\left(\sum_{-N_1 \leq j < 0} \sum_{-N_1 \leq k < 0} a_{i,j,k}^1 z_1^{j/\rho} z_2^{k/\rho}\right) = \text{Re}\left(\sum_{-N_1 \leq j < 0} \sum_{-N_1 \leq k < 0} a_{i,j,k}^2 z_1^{j/\rho} z_2^{k/\rho}\right)$$

Hence, we have

$$\sum_{-N_1 \leq j < 0} \sum_{-N_1 \leq k < 0} a_{i,j,k}^1 z_1^{j/\rho} z_2^{k/\rho} = \sum_{-N_1 \leq j < 0} \sum_{-N_1 \leq k < 0} a_{i,j,k}^2 z_1^{j/\rho} z_2^{k/\rho}.$$

We set

$$f_i^Q := f_{i1}^Q + f_{i2}^Q - \sum_{-N_1 \leq j < 0} \sum_{-N_1 \leq k < 0} a_{i,j,k}^1 z_1^{j/\rho} z_2^{k/\rho}.$$

Then, $\operatorname{Re}(f_i^Q) - g_i^Q$ are bounded on \mathcal{U}_Q .

By shrinking X , we may assume that $X = \varpi(\mathcal{U}_Q)$. Let P be any point of $H \setminus \{(0,0)\}$. Let $\varphi : \Delta \rightarrow X$ be a holomorphic map such that $\varphi(0) = P$. Then, we have $\operatorname{Irr}(\mathbf{E}\varphi^{-1}K) = \{\varphi^* f_i^Q\}$. In particular, f_i^Q are independent of Q . We denote them by f_i .

Let $\varphi : \Delta_\epsilon \rightarrow X$ be any holomorphic map. If $\varphi(0) \neq (0,0)$, we have $\operatorname{Irr}(\mathbf{E}\varphi^{-1}K) = \{\varphi^* f_i\}$, as already mentioned. Suppose that $\varphi(0) = (0,0)$. Let $\varpi_0 : \tilde{\Delta}_\epsilon(0) \rightarrow \Delta_\epsilon$ be the real blowing up. We have the induced map $\tilde{\varphi} : \tilde{\Delta}_\epsilon(0) \rightarrow \tilde{X}(D)$. Note that we have $\dim(\tilde{\varphi}(\varpi_0^{-1}(0)) \cap Z) = 0$ because Z does not contain any positively linear subset. Then, we obtain that $\operatorname{Irr}(\mathbf{E}\varphi^{-1}K) = \{\varphi^* f_i\}$. Thus, we obtain Lemma 6.14. \blacksquare

By Lemma 6.4, Proposition 6.10 and Proposition 6.12, we can assume that the conditions **(C1,2)** are satisfied for $\mathbf{E}\psi_\eta^{-1}K$ at P_η . Then, the claim of Proposition 6.13 follows from Lemma 6.14. \blacksquare

6.6 Resolutions at cross points and generic parts in the surface case

We set $X = \Delta^2$ and $O = (0,0)$. Let $H = \{z_1 = 0\}$ or $\{z_1 = 0\} \cup \{z_2 = 0\}$. Let $K \in \mathbf{E}_\otimes^b(X_H)$.

Proposition 6.15 *Suppose that K satisfies **(A)** at O . Then, we have a meromorphic flat bundle (V, ∇) on (X, H) with an isomorphism $\operatorname{DR}_{X_H}^{\mathbf{E}}(V)[-2] \simeq K$.*

Proof We have a ramified irregular values \mathcal{I} at O . It induces a system of ramified irregular values \mathcal{I} on (X, H) . By using [32], we have a projective morphism $G : X' \rightarrow X$ such that (i) $H' = G^{-1}(H)$ is normal crossing, (ii) G gives an isomorphism $X' \setminus H' \simeq X \setminus H$, (iii) $G^*\mathcal{I}$ is a good system of ramified irregular values. It is enough to construct a meromorphic flat connection (V'_P, ∇_P) with the desired property around any point of $P \in H'$. By Proposition 3.32, it is enough to study the case P is a cross point of H' . Hence, we may assume that \mathcal{I}_O is a good set of ramified irregular values from the beginning. By taking a ramified covering, we may assume that \mathcal{I}_O is a good set of unramified irregular values. By Proposition 6.13, we may also assume that K satisfies **(B)** at O . Let \tilde{L} denote the local system on $\tilde{X}(H)$ induced by $K|_{X \setminus H}$.

Let $H_i = \{z_i = 0\}$. We may assume that the natural map $\mathcal{I} \rightarrow \mathcal{O}_X(*H)/\mathcal{O}_X(*H_2)$ is injective.

According to Proposition 4.13, after shrinking X around O , we may assume that we have a good meromorphic flat bundle (V°, ∇°) on $(X \setminus O, H \setminus O)$ such that $\operatorname{DR}^{\mathbf{E}}(V^\circ, \nabla^\circ)[-2] \simeq K|_{X \setminus O}$. Set $(V_1, \nabla_1) := (V^\circ, \nabla^\circ)|_{X \setminus H_2}$. By Proposition 2.27, we have a unique good meromorphic flat bundle (V, ∇) on (X, H) whose restriction to $X \setminus H_2$ is isomorphic to (V_1, ∇_1) .

Let $\varpi : \tilde{X}(H) \rightarrow X$ be the oriented real blowing up of X along H . Let $\tilde{X}(H) = \tilde{X}(H)^{(0)} \supset \tilde{X}(H)^{(1)} \supset \dots$ be a filtration for $\mathbf{E}\varpi^{-1}K$. Let W be the closure of $\tilde{X}(H)^{(1)} \setminus \partial\tilde{X}(H)$. By the property **(B)**, we may assume to have $\dim_{\mathbb{R}}(W \cap \varpi^{-1}(O)) \leq 1$, and $W \cap \varpi^{-1}(O)$ does not contain any positively linear subset.

Take $Q \in \varpi^{-1}(O) \setminus W$. Then, we have a small neighbourhood \mathcal{U}_Q in $\tilde{X}(H)$ such that $\pi^{-1}(\mathbb{C}_{\mathcal{U}_Q^\circ}) \otimes K$ is controlled by $\operatorname{Re}(f)$ ($f \in \mathcal{I}_O$), i.e.,

$$\pi^{-1}(\mathbb{C}_{\mathcal{U}_Q^\circ}) \otimes K \simeq \bigoplus \mathbb{C}_{\tilde{X}(H)}^{\mathbf{E}} \otimes^+ (\mathbb{C}_{t \geq \operatorname{Re}(f)} \otimes V_f). \quad (25)$$

Here, $\mathcal{U}_Q^\circ = \mathcal{U}_Q \setminus \varpi^{-1}(H)$. We have the canonical filtration on $\pi^{-1}(\mathbb{C}_{\mathcal{U}_Q^\circ}) \otimes K$ as in §3.2.4, which we denote by \mathcal{F}^Q . We may assume that \leq_Q on \mathcal{I}_O is equal to $<$ on $\{\operatorname{Re}(f)|_{\mathcal{U}_Q^\circ} \mid f \in \mathcal{I}_O\}$. The decomposition (25) is compatible with the Stokes filtration of (V_1, ∇_1) at $Q' \in \varpi^{-1}(H_1 \setminus H_2)$. Hence, by the construction in the proof of Proposition 2.27, we obtain that \mathcal{F}^Q is equal to the Stokes filtration of (V, ∇) at Q . Let $Q' \in \varpi^{-1}(H_2 \setminus H_1)$ be contained in \mathcal{U}_Q . Then, the Stokes filtration of (V, ∇) at Q' is equal to the filtration induced by the decomposition (25).

Let $\varphi : \Delta \rightarrow X$ be a holomorphic map such that $\varphi(0) \in H$ and $\varphi(\Delta \setminus \{0\}) \subset X \setminus H$. By the previous consideration, we obtain that the Stokes filtrations of $E\varphi^{-1}K$ and φ^*V are the same at general points of $\varpi_0^{-1}(0)$. Hence, we obtain that $\mathrm{DR}^E \varphi^*(V)[-1] \simeq E\varphi^{-1}K$. Then, the claim of Proposition 6.15 follows from Proposition 3.31. \blacksquare

Corollary 6.16 *For any $K \in E_{\odot}^b(IC_{X(H)})$, there exists a positive integer ℓ_0 such that the following holds for any $\eta \in \{+, -\}^\ell$ if $\ell \geq \ell_0$:*

- *There exists a meromorphic flat bundle V given on a neighbourhood U around P_η , and an isomorphism $E\psi_\eta^{-1}(K)|_U \simeq \mathrm{DR}^E(V)$.*

Proof By Proposition 6.13, we may assume that the conditions (A) and (B) are satisfied. Then, the claim of the corollary follows from Proposition 6.15. \blacksquare

7 Preliminaries for infinite sequence of blowings up of mixed type

We give some preliminaries to deal with infinite sequences of complex blowings up. As mentioned in §5.3.5, we have three classes of infinite sequences called (i), (ii), and (iii). We give a preliminary for the class (i) in §7.1–7.2, and we give a preliminary for the class (ii) in §7.3.

7.1 Vanishing

7.1.1 Statements

Let $\hat{\kappa}$ be a positive real number. Let \mathcal{S} be a subset in $\mathbb{Q} \cap]0, \hat{\kappa}[$ satisfying the following condition.

- For any $\eta \in]0, \hat{\kappa}[$, the intersection $\mathcal{S}_\eta := \mathcal{S} \cap]0, \eta[$ is finite.

We set $T_m(\mathcal{S}) := \mathbb{Z}_{\geq 0} \times \mathcal{S}^m$ ($m \geq 1$) and $T_0(\mathcal{S}) = \mathbb{Z}_{\geq 0}$. We put $T_+(\mathcal{S}) := \coprod_{m \geq 1} T_m(\mathcal{S})$ and $T(\mathcal{S}) := T_0(\mathcal{S}) \cup T_+(\mathcal{S})$. For any element $\mathbf{s} = (s_1, \dots, s_m) \in \mathcal{S}^m$, the number m is denoted by $|\mathbf{s}|$.

Let e be a positive integer. For any $L \geq 0$, we set

$$T_+(\mathcal{S}, L) := \left\{ (i, s_1, \dots, s_m) \in T_+(\mathcal{S}, L) \mid \frac{i}{e} + \sum_{p=1}^m s_p = L \right\},$$

and $T_0(\mathcal{S}, L) := \{i \in T_0(\mathcal{S}) \mid i/e = L\}$. We set $T(\mathcal{S}, L) := T_+(\mathcal{S}, L) \cup T_0(\mathcal{S}, L)$.

Let \mathcal{V}_j ($j = 1, 2, \dots$) be a sequence of neighbourhoods of 0 in \mathbb{C} such that $\mathcal{V}_j \supset \mathcal{V}_{j+1}$ for any j . Let $\mathcal{O}(\mathcal{V}_j)$ be the space of holomorphic functions on \mathcal{V}_j . Let $\varphi_j : \mathbb{R}_{>0} \rightarrow \mathcal{O}(\mathcal{V}_j)$ ($j = 1, 2, \dots$) be maps with the following property:

- $\varphi_j(t) = 0$ unless $t \in \mathcal{S}$.
- For each $\eta \in \mathcal{S}$, there exists $j_0(\eta)$ such that $\varphi_{j_1}(\eta) = \varphi_{j_2}(\eta)$ for any $j_1, j_2 \geq j_0(\eta)$. Moreover, $\varphi_j(\eta)$ are constant functions if $j \geq j_0(\eta)$.

We set $\varphi_\infty(\eta) := \lim_{j \rightarrow \infty} \varphi_j(\eta)$ in the stalk $\mathcal{O}_{\mathbb{C}, 0}$ of the sheaf of holomorphic functions at 0. Because $\varphi_\infty(\eta)$ are the germs of constant functions, we may also regard $\varphi_\infty(\eta)$ as complex numbers.

For any $\eta \in \mathbb{R}_{>0}$, we have the following real analytic functions $R_j(\eta) : \mathbb{R} \times \mathcal{V}_j \rightarrow \mathbb{R}$ and $I_j(\eta) : \mathbb{R} \times \mathcal{V}_j \rightarrow \mathbb{R}$ ($j = 1, 2, \dots$) given by

$$R_j(\eta)(\phi) := \mathrm{Re}(\varphi_j(\eta)e^{\sqrt{-1}\eta\phi}), \quad I_j(\eta)(\phi) := \mathrm{Im}(\varphi_j(\eta)e^{\sqrt{-1}\eta\phi}).$$

For any point $\phi_0 \in \mathbb{R}$, the induced germs of analytic functions at $(\phi_0, 0) \in \mathbb{R} \times \mathbb{C}$ are also denoted by the same notation. For each η , we have j_0 such that $R_{j_1}(\eta) = R_{j_2}(\eta)$ and $I_{j_1}(\eta) = I_{j_2}(\eta)$ for any $j_1, j_2 \geq j_0$ as germs of

analytic functions at $(\phi_0, 0)$. We set $R_\infty(\eta) := \lim_{j \rightarrow \infty} R_j(\eta)$ and $I_\infty(\eta) := \lim_{j \rightarrow \infty} I_j(\eta)$ in the space of germs of analytic functions at $(\phi_0, 0)$. We have

$$R_\infty(\eta)(\phi) := \operatorname{Re}(\varphi_\infty(\eta)e^{\sqrt{-1}\eta\phi}), \quad I_\infty(\eta)(\phi) := \operatorname{Im}(\varphi_\infty(\eta)e^{\sqrt{-1}\eta\phi}).$$

We may also regard them as analytic functions on \mathbb{R} , or germs of analytic functions at $\phi_0 \in \mathbb{R}$.

Let \mathcal{I} be an interval. Let $f_{i,c_1,c_2} : \mathcal{I} \rightarrow \mathbb{R}$ ($(i, c_1, c_2) \in \mathbb{Z}_{\geq 0}^3$) be a family of real analytic functions. We assume that $f_{0,0,0}$ is constantly 0.

Let L be a non-negative real number. For any pair $(\ell_1, \ell_2, m) \in \mathbb{Z}_{\geq 0}^3$ satisfying $m = \ell_1 + \ell_2$, we have the following functions on $\mathcal{I} \times \mathcal{V}_j$ ($j = 1, 2, \dots$):

$$\begin{aligned} A_j^{(m)}(L, (\ell_1, \ell_2)) := & \sum_{(i, \zeta) \in T_+(\mathcal{S}, L)} \sum_{\substack{(c_1, c_2) \in \mathbb{Z}_{\geq 0}^2 \\ c_1 + c_2 = |\zeta| + m}} f_{i, c_1, c_2} \frac{c_1! c_2!}{(c_1 - \ell_1)!(c_2 - \ell_2)!} \prod_{q=1}^{c_1 - \ell_1} R_j(\zeta_q) \prod_{q=c_1 - \ell_1 + 1}^{|\zeta|} I_j(\zeta_q) \\ & + \sum_{i \in T_0(\mathcal{S}, L)} \ell_1! \ell_2! f_{i, \ell_1, \ell_2}. \end{aligned} \quad (26)$$

Here, we use the convention $\prod_{q=c}^{c-1} a_q := 1$. We fix any point $\phi_0 \in \mathcal{I}$, and the induced germs of real analytic functions at $(\phi_0, 0) \in \mathcal{I} \times \mathbb{C}$ are also denoted by $A_j^{(m)}(L, (\ell_1, \ell_2))$.

For each pair of L and m , the set $\{(i, \zeta, c_1, c_2) \mid (i, \zeta) \in T_L(\mathcal{S}, L), c_1 + c_2 = |\zeta| + m\}$ is finite. Hence, we have j_0 depending on (L, m) such that $A_{j_1}^{(m)}(L, \ell_1, \ell_2) = A_{j_2}^{(m)}(L, \ell_1, \ell_2)$ for any $j_1, j_2 \geq j_0$ as germs of analytic functions at $(\phi_0, 0) \in \mathcal{I} \times \mathbb{C}$. We set $A_\infty^{(m)}(L, \ell_1, \ell_2) := \lim_{j \rightarrow \infty} A_j^{(m)}(L, \ell_1, \ell_2)$ in the space of germs of analytic functions at $(\phi_0, 0)$. We have the following expression:

$$\begin{aligned} A_\infty^{(m)}(L, (\ell_1, \ell_2)) := & \sum_{(i, \zeta) \in T_+(\mathcal{S}, L)} \sum_{\substack{(c_1, c_2) \in \mathbb{Z}_{\geq 0}^2 \\ c_1 + c_2 = |\zeta| + m}} f_{i, c_1, c_2} \frac{c_1! c_2!}{(c_1 - \ell_1)!(c_2 - \ell_2)!} \prod_{q=1}^{c_1 - \ell_1} R_\infty(\zeta_q) \prod_{q=c_1 - \ell_1 + 1}^{|\zeta|} I_\infty(\zeta_q) \\ & + \sum_{i \in T_0(\mathcal{S}, L)} \ell_1! \ell_2! f_{i, \ell_1, \ell_2}. \end{aligned} \quad (27)$$

We may also regard them as real analytic functions on \mathbb{R} . When $(m, \ell_1, \ell_2) = (0, 0, 0)$, we use the notation $A_j^{(0)}(L)$ and $A_\infty^{(0)}(L)$, instead of $A_j^{(0)}((L), (0, 0))$ and $A_\infty^{(0)}((L), (0, 0))$. We shall prove the following propositions.

Proposition 7.1 *Suppose that we are given $L_1 \geq 0$ such that $A_\infty^{(0)}(L) = 0$ for any $L < L_1$. Then, the following holds for any $d \geq 1$:*

- We have $A_\infty^{(d)}(L - d\hat{\kappa}, (\ell_1, \ell_2)) = 0$ for any $L \leq L_1$, and any $(\ell_1, \ell_2) \in \mathbb{Z}_{\geq 0}^2$ satisfying $\ell_1 + \ell_2 = d \geq 1$.
- There exists j_0 such that we have $A_j^{(d)}(L - d\hat{\kappa}, (\ell_1, \ell_2)) = 0$ for any $L \leq L_1$, any $j \geq j_0$, and any $(\ell_1, \ell_2) \in \mathbb{Z}_{\geq 0}^2$ satisfying $\ell_1 + \ell_2 = d \geq 1$.

Proposition 7.2 *Suppose that $A_\infty^{(0)}(L) = 0$ for any $L \geq 0$. Then, we have $f_{i, c_1, c_2} = 0$ for any (i, c_1, c_2) .*

7.1.2 Preliminary

As a preliminary, we consider the following condition for subsets $S \subset \mathbb{R}_{\geq 0}$:

(P) For any $a \in \mathbb{R}_{\geq 0}$, there exists $\epsilon > 0$ such that $S \cap]a, a + \epsilon[$ is empty.

Lemma 7.3 *Let S be a subset in $\mathbb{R}_{\geq 0}$ satisfying the condition (P). Let s_i ($i = 1, 2, \dots$) be a decreasing sequence in S . Then, there exists i_0 such that $s_i = s_{i_0}$ for any $i \geq i_0$.*

Proof We have $s_\infty = \lim_{i \rightarrow \infty} s_i$. We have $\epsilon > 0$ such that $]s_\infty, s_\infty + \epsilon[\cap S = \emptyset$. Then, we have $s_i = s_\infty$ for any sufficiently large i . ■

For any subset $S \subset \mathbb{R}_{\geq 0}$, let $\text{Accum}(S)$ denote the set of accumulation points in S , i.e., $P \in \mathbb{R}_{\geq 0}$ is contained in $\text{Accum}(S)$ if and only if P is contained in the closure of $S \setminus \{P\}$.

Lemma 7.4 *Let S_i ($i = 1, 2$) be subsets in $\mathbb{R}_{\geq 0}$ satisfying the condition (P). Then, $S_1 \cup S_2$ also satisfies the condition (P). We also have $\text{Accum}(S_1 \cup S_2) = \text{Accum}(S_1) \cup \text{Accum}(S_2)$.*

Proof Take any $a \in \mathbb{R}_{\geq 0}$. We have $\epsilon_i > 0$ such that $]a, a + \epsilon_i[\cap S_i = \emptyset$. Set $\epsilon_0 := \min\{\epsilon_1, \epsilon_2\}$. Then, we have $]a, a + \epsilon_0[\cap (S_1 \cup S_2) = \emptyset$. Hence, $S_1 \cup S_2$ also satisfies (P).

We clearly have $\text{Accum}(S_1 \cup S_2) \supset \text{Accum}(S_1) \cup \text{Accum}(S_2)$. Let $s \in \text{Accum}(S_1 \cup S_2)$. We have a sequence $s_i \in (S_1 \cup S_2) \setminus \{s\}$ such that $\lim s_i = s$. One of S_1 or S_2 contains infinite subsequence of s_i . Hence, $s \in \text{Accum}(S_1) \cup \text{Accum}(S_2)$. ■

Lemma 7.5 *Let $S_i \subset \mathbb{R}_{\geq 0}$ ($i = 1, 2, \dots$) be subsets satisfying the condition (P). Suppose that*

$$\liminf_{i \rightarrow \infty} (S_i \setminus \{0\}) = \infty.$$

Then, $\bigcup_{i \geq 1} S_i$ also satisfies the condition (P). We also have $\text{Accum}\left(\bigcup_{i \geq 1} S_i\right) = \bigcup_{i \geq 1} \text{Accum}(S_i)$.

Proof Take any $a \in \mathbb{R}_{\geq 0}$ and $\epsilon_0 > 0$. We have i_0 such that $]a, a + \epsilon_0[\cap S_i = \emptyset$ for any $i \geq i_0$. We have $]a, a + \epsilon_0[\cap \left(\bigcup_{i \geq 1} S_i\right) =]a, a + \epsilon_0[\cap \left(\bigcup_{i=1}^{i_0-1} S_i\right)$. Because $\bigcup_{i=1}^{i_0-1} S_i$ satisfies the condition (P), we obtain the first claim of the lemma. We clearly have $\text{Accum}\left(\bigcup_{i \geq 1} S_i\right) \supset \bigcup_{i \geq 1} \text{Accum}(S_i)$. Take $s \in \text{Accum}\left(\bigcup_{i \geq 1} S_i\right)$ and $\epsilon > 0$. We have i_0 such that $]s - \epsilon, s + \epsilon[\cap \text{Accum}\left(\bigcup_{i \geq 1} S_i\right) =]s - \epsilon, s + \epsilon[\cap \text{Accum}\left(\bigcup_{i=1}^{i_0} S_i\right)$. Then, we obtain the second claim by using the argument in the proof of Lemma 7.4. ■

The following lemma is clear.

Lemma 7.6 *Let S be a subset in $\mathbb{R}_{\geq 0}$ satisfying the condition (P). Then, any subset $S' \subset S$ also satisfies the condition (P).* ■

Let $S_i \subset \mathbb{R}_{\geq 0}$ ($i = 1, 2$) be subsets satisfying the following conditions.

- We have $\widehat{\kappa} > 0$ for which S_1 is contained in $]0, \widehat{\kappa}[$. Moreover, for any $\eta < \widehat{\kappa}$, the intersection $S_1 \cap]0, \eta]$ is finite.
- S_2 satisfies the condition (P).

We set $S_1 + S_2 := \{t \in \mathbb{R}_{\geq 0} \mid \exists (a_1, a_2) \in S_1 \times S_2, a_1 + a_2 = t\}$.

Lemma 7.7 *$S_1 + S_2$ also satisfies the condition (P). Moreover, for any $c \in S_1 + S_2$, the set $\{(c_1, c_2) \in S_1 \times S_2 \mid c_1 + c_2 = c\}$ is finite.*

Proof Take any $a \in \mathbb{R}_{\geq 0}$. We set $\mathbb{R}_{>a} := \{t \in \mathbb{R} \mid t > a\}$. Let t_i be a decreasing sequence in $S_1 + S_2$ such that $\lim t_i = a$. We have $(b_{i,1}, b_{i,2}) \in S_1 \times S_2$ such that $t_i = b_{i,1} + b_{i,2}$. By taking a sub-sequence, we may assume that $b_{i,1} \leq b_{i+1,1}$ for any i . Then, $b_{i,2} = t_i - b_{i,1}$ is decreasing. Hence, there exists i_0 such that $b_{i,2} = b_{i_0,2}$ for any $i \geq i_0$. Because t_i is decreasing and $b_{i,1}$ is increasing, we have i_1 such that $t_i = t_{i_1}$ and $b_{i,1} = b_{i_1,1}$ for any $i \geq i_1$, i.e., $t_i = a$ for any $i \geq i_1$. Hence, we obtain the first claim.

Let $c \in S_1 + S_2$. If the set $\{(c_1, c_2) \in S_1 \times S_2 \mid c_1 + c_2 = c\}$ is infinite, we can take an infinite sequence $(c_{i,1}, c_{i,2})$ in $S_1 \times S_2$ such that (i) $c_{i,1} + c_{i,2} = c$, (ii) $c_{i,1}$ is increasing. Then, $c_{i,2}$ is decreasing, and hence we have i_0 such that $c_{i,2} = c_{i_0,2}$ for any $i \geq i_0$. It contradicts with the assumption that the sequence $(c_{i,1}, c_{i,2})$ is infinite. ■

Let \overline{S}_i ($i = 1, 2$) denote the closure of the above sets S_i in $\mathbb{R}_{\geq 0}$. Let $\text{Accum}'(S_1 + S_2)$ denote the image of the map $(\text{Accum}(S_1) \times S_2) \cup (\overline{S}_1 \times \text{Accum}(S_2)) \rightarrow \mathbb{R}_{\geq 0}$ given by $(c_1, c_2) \mapsto c_1 + c_2$. Note that $\text{Accum}(S_1) = \{\widehat{\kappa}\}$.

Lemma 7.8 *We have $\text{Accum}(S_1 + S_2) = \text{Accum}'(S_1 + S_2)$.*

Proof We clearly have $\text{Accum}(S_1 + S_2) \supset \text{Accum}'(S_1 + S_2)$. Let us prove $\text{Accum}(S_1 + S_2) \subset \text{Accum}'(S_1 + S_2)$. Take $c \in \text{Accum}(S_1 + S_2)$. We have an increasing sequence c_i in $(S_1 + S_2) \setminus \{c_i\}$ such that $\lim_{i \rightarrow \infty} c_i = c$. We have $(c_i^{(1)}, c_i^{(2)}) \in S_1 \times S_2$ such that $c_i^{(1)} + c_i^{(2)} = c_i$. By taking a sub-sequence, we may assume that $c_i^{(1)}$ is increasing. Suppose that we have i_0 such that $c_i^{(1)} = c_{i_0}^{(1)}$ for any $i \geq i_0$. The sequence $c_i^{(2)} = c_i - c_i^{(1)}$ is convergent to $c - c_{i_0}^{(1)}$. Hence, we have $c - c_{i_0}^{(1)} \in \text{Accum}(S_2)$, and $c \in \text{Accum}'(S_1 \times S_2)$. Suppose that $\lim_{i \rightarrow \infty} c_i^{(1)} = \widehat{\kappa}$. The sequence $c_i^{(2)} = c_i - c_i^{(1)}$ is convergent to $c - \widehat{\kappa}$. Hence, we have $c - \widehat{\kappa} \in \overline{S_2}$, and hence we have $c \in \text{Accum}'(S_1 \times S_2)$. ■

Let S_1 be the above set. For $\ell \geq 1$, let $\sum^\ell S_1$ denote the image of the map $S_1^\ell \rightarrow \mathbb{R}_{\geq 0}$ given by $(s_1, \dots, s_\ell) \mapsto \sum_{i=1}^\ell s_i$. We formally set $\sum^0 S_1 := \{0\}$.

Corollary 7.9 *The set $\sum^\ell S_1$ satisfies the condition (P). We have*

$$\text{Accum}\left(\sum^\ell S_1\right) = \bigcup_{1 \leq m \leq \ell} \left(\{m\widehat{\kappa}\} + \sum^{\ell-m} S_1\right).$$

Proof We obtain the first claim from Lemma 7.7 and an induction. Let us study the second claim. We use an induction on ℓ . The claim is clear in the cases $\ell = 0, 1$. According to Lemma 7.8, we have $\text{Accum}(\sum^\ell S_1) = (\{\widehat{\kappa}\} + \overline{\sum^{\ell-1} S_1}) \cup (\overline{S_1} + \text{Accum}(\sum^{\ell-1} S_1))$. By the assumption of the induction, we have $\text{Accum}(\sum^{\ell-1} S_1) = \bigcup_{1 \leq m \leq \ell-1} (\{m\widehat{\kappa}\} + \sum^{\ell-1-m} S_1)$. We have $\overline{S_1} = S_1 \sqcup \{\widehat{\kappa}\}$. Hence, we have

$$\begin{aligned} \overline{S_1} + \text{Accum}\left(\sum^{\ell-1} S_1\right) &= \bigcup_{1 \leq m \leq \ell-1} \left(\{(m+1)\widehat{\kappa}\} + \sum^{\ell-1-m} S_1\right) \cup \bigcup_{1 \leq m \leq \ell-1} \left(\{m\widehat{\kappa}\} + \sum^{\ell-m} S_1\right) \\ &= \bigcup_{1 \leq m \leq \ell} \left(\{m\widehat{\kappa}\} + \sum^{\ell-m} S_1\right). \end{aligned} \quad (28)$$

We have $\{\widehat{\kappa}\} + \overline{\sum^{\ell-1} S_1} = (\{\widehat{\kappa}\} + \sum^{\ell-1} S_1) \cup (\{\widehat{\kappa}\} + \text{Accum}(\sum^{\ell-1} S_1))$. Hence, we obtain the claim in the case ℓ . ■

Corollary 7.10 *Let e be a positive integer. The set $\bigcup_{i \geq 0} \bigcup_{\ell \geq 0} (\{i/e\} + \sum^\ell S_1)$ satisfies the condition (P). We also have*

$$\text{Accum}\left(\bigcup_{i \geq 0} \bigcup_{\ell \geq 1} (\{i/e\} + \sum^\ell S_1)\right) = \bigcup_{i \geq 0} \bigcup_{m \geq 1} \bigcup_{\ell \geq 0} \left(\{i/e + m\widehat{\kappa}\} + \sum^\ell S_1\right).$$

Proof Let $\beta_0 > 0$ be the infimum of S_1 . The infimum of $\{i/e\} + \sum^\ell S_1$ is $i/e + \ell\beta_0$. The claim follows from Lemma 7.5 and Corollary 7.9. ■

7.1.3 Accumulation

We return to the situation in §7.1.1. We set $\mathcal{T}(\mathcal{S}) := \{i/e + \sum \zeta_p \mid (i, \zeta) \in T_+(\mathcal{S})\} \cup \frac{1}{e}\mathbb{Z}_{\geq 0}$. According to Corollary 7.10, the set $\mathcal{T}(\mathcal{S})$ satisfies the condition (P), and we have

$$\text{Accum}(\mathcal{T}(\mathcal{S})) = \bigcup_{m \geq 1} \left(\{m\widehat{\kappa}\} + \mathcal{T}(\mathcal{S})\right).$$

Similarly, we put $\mathcal{T}(\mathcal{S}_\eta) := \{i/e + \sum_{j=1}^m \zeta_j \mid (i, \zeta) \in T_+(\mathcal{S}_\eta)\} \cup \frac{1}{e}\mathbb{Z}_{\geq 0}$ for any $\eta \in]0, \widehat{\kappa}[$. The set $\mathcal{T}(\mathcal{S}_\eta)$ is discrete.

Fix $L \in \mathbb{R}_{\geq 0}$. We can take $\delta > 0$ such that the following holds for any $m \geq 0$:

$$]L - m\widehat{\kappa}, L - m\widehat{\kappa} + \delta[\cap \mathcal{T}(\mathcal{S}) = \emptyset.$$

Because the set $\bigcup_{m \leq 0} T(\mathcal{S}, L - m\hat{\kappa})$ is finite, we can take $\eta_0 \in]\hat{\kappa} - \delta, \hat{\kappa}[$ such that

$$\bigcup_{m \geq 0} T(\mathcal{S}, L - m\hat{\kappa}) = \bigcup_{m \geq 0} T(\mathcal{S}_{\eta_0}, L - m\hat{\kappa}).$$

Lemma 7.11 *For any $\eta > \eta_0$, there exists $\epsilon > 0$ such that the following holds:*

- *If $(i, \zeta_1, \dots, \zeta_q) \in T(\mathcal{S})$ satisfies $L - m\hat{\kappa} - \epsilon < i/e + \sum \zeta_p < L - m\hat{\kappa}$ for some $m \geq 0$, then we have $\max\{\zeta_p\} > \eta$.*

Proof Because the set $\mathcal{T}(\mathcal{S}_\eta)$ is discrete, we can take $\epsilon > 0$ satisfying $\epsilon < L - m\hat{\kappa} - \max\{\nu \in \mathcal{T}(\mathcal{S}_\eta) \mid \nu < L - m\hat{\kappa}\}$ for any $m \geq 0$. Then, the condition is satisfied. \blacksquare

Lemma 7.12 *If $(i, \zeta_1, \dots, \zeta_q) \in T(\mathcal{S})$ satisfies $i/e + \sum \zeta_p = L - m\hat{\kappa}$ for some $m \geq 0$, then $\max\{\zeta_p\} < \eta_0$.*

Proof Suppose that $(i, \zeta_1, \dots, \zeta_q) \in T(\mathcal{S})$ satisfies $i/e + \sum \zeta_p = L - m\hat{\kappa}$ for some $m \geq 0$. We assume that $\zeta_1 \geq \eta_0$, and we shall derive a contradiction. We have $0 < \hat{\kappa} - \zeta_1 \leq \hat{\kappa} - \eta_0 < \delta$. Then, we have $L - (m+1)\hat{\kappa} < i/e + \sum_{p=2}^q \zeta_p < L - (m+1)\hat{\kappa} + \delta$. Such $(i, \zeta_2, \dots, \zeta_q)$ cannot exist by our choice of δ . \blacksquare

Lemma 7.13 *For any $\eta \in]\eta_0, \hat{\kappa}[$ we can take $\epsilon > 0$ such that the following holds:*

- *If $(i, \zeta_1, \dots, \zeta_q) \in T(\mathcal{S})$ satisfies (i) $L - \epsilon < i/e + \sum \zeta_p < L$, (ii) $\zeta_1 \geq \dots \geq \zeta_q$, then we have a unique positive integer ℓ for which the following holds:*

$$\zeta_1 \geq \dots \geq \zeta_\ell > \eta, \quad \zeta_{\ell+1} + \dots + \zeta_q = L - \ell\hat{\kappa}.$$

Note that we have $\zeta_p < \eta_0$ ($p = \ell + 1, \dots, q$) by Lemma 7.12.

Proof We set $\gamma(L) := \#\{m \geq 1 \mid L - m\hat{\kappa} \in \overline{\mathcal{T}(\mathcal{S})}\}$. We use an induction on $\gamma(L)$.

Suppose $\gamma(L) = 0$. For any $\eta > \eta_0$, we take $\epsilon > 0$ as in Lemma 7.11. We have $\mathcal{T}(\mathcal{S}) \cap]L - \hat{\kappa} - \epsilon, L - \hat{\kappa} + \delta[= \emptyset$. If $(i, \zeta_1, \dots, \zeta_q)$ satisfies $L - \epsilon < i/e + \sum \zeta_j < L$ and $\zeta_1 \geq \dots \geq \zeta_q$, then we have $\zeta_1 > \eta$. We should have

$$L - \hat{\kappa} - \epsilon < i/e + \sum_{j \geq 2} \zeta_j < L - \hat{\kappa} + (\hat{\kappa} - \eta) < L - \hat{\kappa} + \delta.$$

Hence, we do not have such $(i, \zeta_1, \dots, \zeta_q)$. Thus, we are done in the case $\gamma(L) = 0$.

Suppose that we have already proved the case $\gamma(L) \leq n$, and we shall prove the claim in the case $\gamma(L) = n+1$. Note that $\gamma(L - \hat{\kappa}) = n$ if $\gamma(L) = n+1$. Indeed, because $n+1 \geq 1$, we have $m \geq 1$ such that $L - m\hat{\kappa} \in \overline{\mathcal{T}(\mathcal{S})}$. Then, it is easy to observe $L - \hat{\kappa} = L - m\hat{\kappa} + (m-1)\hat{\kappa} \in \overline{\mathcal{T}(\mathcal{S})}$. Hence, $\gamma(L - \hat{\kappa}) = \gamma(L) - 1 = n$.

Let $\eta > \eta_0$. By the assumption of the induction, we have a positive constant ϵ_1 for the pair of $L - \hat{\kappa}$ and η satisfying the property in the statement of Lemma 7.13. We also have a positive constant ϵ for the pair of $L - \hat{\kappa}$ and η as in Lemma 7.11. We take $0 < \epsilon_2 < \min\{\epsilon_1, \epsilon\}$. Suppose that $(i, \zeta_1, \dots, \zeta_q)$ satisfies $L - \epsilon_2 < i/e + \sum \zeta_j < L$ and $\zeta_1 \geq \dots \geq \zeta_q$. We have $\hat{\kappa} > \zeta_1 > \eta$, and hence

$$L - \hat{\kappa} - \epsilon_2 < i/e + \sum_{j \geq 2} \zeta_j < (L - \hat{\kappa}) + (\hat{\kappa} - \zeta_1) < L - \hat{\kappa} + \delta.$$

By our choice of δ , we have $i/e + \sum_{j \geq 2} \zeta_j \leq L - \hat{\kappa}$. It implies either one of the following:

- (i) $i/e + \sum_{j \geq 2} \zeta_j = L - \hat{\kappa}$.
- (ii) $L - \hat{\kappa} - \epsilon_2 < i/e + \sum_{j \geq 2} \zeta_j < L - \hat{\kappa}$.

If (i) occurs, we are done by Lemma 7.12. If (ii) occurs, we may apply the hypothesis of the induction. \blacksquare

Let $L \in \mathbb{R}_{\geq 0}$. Suppose that the set $\{m \in \mathbb{Z}_{\geq 0} \mid L - m\hat{\kappa} \in \overline{\mathcal{T}(\mathcal{S})}\}$ is not empty. Let m_0 be the maximum element of the set.

Lemma 7.14 *$L - m_0\hat{\kappa}$ is an isolated point in $\mathcal{T}(\mathcal{S})$.*

Proof By the assumption, we have $L - m_0\hat{\kappa} \in \overline{\mathcal{T}(\mathcal{S})}$. If $L - m_0\hat{\kappa}$ is an accumulation point of $\mathcal{T}(\mathcal{S})$, it is contained in $\bigcup_{m \geq 1} (\{m\hat{\kappa}\} + \mathcal{T}(\mathcal{S}))$, i.e., $L - m_0\hat{\kappa} = m\hat{\kappa} + s$ for some $m \geq 1$ and $s \in \mathcal{T}(\mathcal{S})$. We have $L - (m_0 + m)\hat{\kappa} = s \in \mathcal{T}(\mathcal{S})$ which contradicts with the choice of m_0 . Hence, $L - m_0\hat{\kappa}$ is an isolated point of $\mathcal{T}(\mathcal{S})$. \blacksquare

7.1.4 A formula

For any $L \geq 0$, $\eta \in]0, \widehat{\kappa}[$ and $d \in \mathbb{Z}_{>0}$, we set $\mathcal{U}(d, \eta, L) := \{(\zeta_1, \dots, \zeta_d) \in \mathcal{S}^d \mid \sum_{i=1}^d \zeta_i = L, \zeta_i > \eta\}$. For any $(d_1, d_2, d) \in \mathbb{Z}_{\geq 0}^3$ such that $d_1 + d_2 = d$, and for any $j = 1, \dots, \infty$, we set

$$B_j(L, \eta, (d_1, d_2)) := \sum_{\zeta \in \mathcal{U}(d, \eta, L)} \frac{1}{d_1! d_2!} \prod_{q=1}^{d_1} R_j(\zeta_q) \prod_{q=d_1+1}^d I_j(\zeta_q).$$

Corollary 7.15 *For any $L \in \mathbb{R}_{\geq 0}$, we have $\epsilon_0 > 0$ and $0 < \eta_0 < \widehat{\kappa}$ such that the following holds for any L' with $L - \epsilon_0 < L' < L$:*

$$A_j^{(m)}(L', (b_1, b_2)) = \sum_{d \geq 1} \sum_{\substack{d_1 + d_2 = d \\ d_i \geq 0}} A_j^{(m+d)}(L - d\widehat{\kappa}, (d_1, d_2) + (b_1, b_2)) \cdot B_j(L' - (L - d\widehat{\kappa}), \eta_0, (d_1, d_2)). \quad (29)$$

Proof It follows from Lemma 7.12 and Lemma 7.13. ■

7.1.5 Vanishing at ∞

We fix $L > 0$.

Lemma 7.16 *Suppose that $A_\infty^{(0)}(L') = 0$ for any $L' < L$. Then, we have $A_\infty^{(m)}(L' - m\widehat{\kappa}, (b_1, b_2)) = 0$ for any $L' \leq L$ and $(b_1, b_2, m) \in \mathbb{Z}_{\geq 0}^2 \times \mathbb{Z}_{\geq 1}$ satisfying $b_1 + b_2 = m$.*

Proof We set $\widetilde{\mathcal{T}}(\mathcal{S}) := \bigcup_{m \geq 0} (\{m\widehat{\kappa}\} + \mathcal{T}(\mathcal{S}))$. The set $\widetilde{\mathcal{T}}(\mathcal{S})$ satisfies the condition **(P)**.

If $L' \notin \widetilde{\mathcal{T}}(\mathcal{S})$, then we have $A_\infty^{(m)}(L' - m\widehat{\kappa}) = 0$ for any $m \geq 0$. Hence, it is enough to prove the vanishing $A_\infty^{(m)}(L' - m\widehat{\kappa}) = 0$ ($m \geq 1$) under the following assumption:

(Q) $A_\infty^{(m)}(L'' - m\widehat{\kappa}) = 0$ ($m \geq 0$) for any $L'' < L'$.

It is enough to consider the case that the set $\{m \in \mathbb{Z}_{\geq 1} \mid L' - m\widehat{\kappa} \in \overline{\mathcal{T}(\mathcal{S})}\}$ is not empty. Let m_0 be the maximum element. By the choice of m_0 , we have $A_\infty^{(m)}(L' - m\widehat{\kappa}, (b_1, b_2)) = 0$ for any $m > m_0$ and for any $(b_1, b_2) \in \mathbb{Z}_{\geq 0}^2$ satisfying $b_1 + b_2 = m$. Let us prove the vanishing for any m by a descending induction.

According to Lemma 7.14, $L' - m_0\widehat{\kappa}$ is an isolated point of $\mathcal{T}(\mathcal{S})$. By the assumption **(Q)**, we have

$$A^{(m_0-1)}(L' - m_0\widehat{\kappa} + \eta, (b_1, b_2)) = A^{(m_0-1)}(L' + \eta - \widehat{\kappa} - (m_0 - 1)\widehat{\kappa}, (b_1, b_2)) = 0$$

for any $\eta \in \mathcal{S}$ and for any $(b_1, b_2) \in \mathbb{Z}_{\geq 0}^2$ with $b_1 + b_2 = m_0 - 1$. If η is sufficiently close to $\widehat{\kappa}$, we obtain the following equality from Corollary 7.15:

$$0 = A_\infty^{(m_0-1)}(L' - m_0\widehat{\kappa} + \eta, (b_1, b_2)) = \sum_{c_1 + c_2 = 1} A_\infty^{(m_0)}(L' - m_0\widehat{\kappa}, (c_1, c_2) + (b_1, b_2)) \cdot B_\infty(\eta, \eta_0, (c_1, c_2)).$$

Then, we obtain $A_\infty^{(m_0)}(L' - m_0\widehat{\kappa}, (d_1, d_2)) = 0$ for any $(d_1, d_2) \in \mathbb{Z}_{\geq 0}^2$ satisfying $d_1 + d_2 = m_0$.

Suppose that we have already known the vanishing $A_\infty^{(m)}(L' - m\widehat{\kappa}, (b_1, b_2)) = 0$ for any $(b_1, b_2) \in \mathbb{Z}_{\geq 0}^2$ satisfying $b_1 + b_2 = m$ for any $m > m_1 \geq 1$, and let us prove the vanishing in the case m_1 . By the assumption **(Q)**, we have $A_\infty^{(m_1-1)}(L' - m_1\widehat{\kappa} + \eta, (b_1, b_2)) = 0$ for any $\eta \in \mathcal{S}$ and for any $(b_1, b_2) \in \mathbb{Z}_{\geq 0}^2$ satisfying $b_1 + b_2 = m_1 - 1$. By Corollary 7.15, we have the following equality:

$$0 = A_\infty^{(m_1-1)}(L' - m_1\widehat{\kappa} + \eta, (b_1, b_2)) = \sum_{d \geq 0} \sum_{c_1 + c_2 = d+1} A_\infty^{(m_1+d)}(L - (m_1 + d)\widehat{\kappa}, (c_1, c_2) + (b_1, b_2)) \cdot B_\infty(\eta + d\widehat{\kappa}, \eta_0, (c_1, c_2)). \quad (30)$$

By using the assumption of the induction, we can rewrite it as follows:

$$0 = \sum_{c_1 + c_2 = 1} A_\infty^{(m_1)}(L_1 - m_1\widehat{\kappa}, (c_1, c_2) + (b_1, b_2)) \cdot B_\infty(\eta, \eta_0, (c_1, c_2)).$$

Hence, we obtain $A_\infty^{(m_1)}(L_1 - m_1\widehat{\kappa}, (b_1, b_2)) = 0$ for any $(d_1, d_2) \in \mathbb{Z}_{\geq 0}^2$ satisfying $d_1 + d_2 = m_1$. Thus, we obtain Lemma 7.16. ■

7.1.6 Proof of Proposition 7.1

Let us prove Proposition 7.1. We have already known the first claim by Lemma 7.16.

Lemma 7.17 *For any $L' \leq L$, there exists $j_0(L') \in \mathbb{Z}_{\geq 1}$ and $\epsilon(L') > 0$ such that $A_j^{(m)}(L'_1 - m\hat{\kappa}, (c_1, c_2)) = 0$ for any $L'_1 \in]L' - \epsilon(L'), L' + \epsilon(L')[\cap [0, L]$, any $j \geq j_0(L')$, and any $(c_1, c_2, m) \in \mathbb{Z}_{\geq 0}^2 \times \mathbb{Z}_{\geq 1}$ satisfying $c_1 + c_2 = m$.*

Proof It is enough to consider the claim for $L'_1 \in]L' - \epsilon(L'), L' \cap [0, L]$. By Corollary 7.15, if $\epsilon(L')$ is sufficiently small, we have the following:

$$A_j^{(m)}(L'_1, (b_1, b_2)) = \sum_{d \geq 1} \sum_{c_1 + c_2 = d} A_j^{(m+d)}(L' - d\hat{\kappa}, (c_1, c_2) + (b_1, b_2)) \cdot B_j(L'_1 - L' + d\hat{\kappa}, \eta_0, (c_1, c_2)).$$

If $j_0(L')$ is sufficiently large, we have $A_j^{(m+d)}(L' - d\hat{\kappa}, (c_1, c_2) + (b_1, b_2)) = 0$ for any $j \geq j_0(L')$. Then, we obtain the claim of Lemma 7.17. \blacksquare

Then, by using the compactness of the interval $[0, L]$, we obtain the claim of Proposition 7.1. \blacksquare

7.1.7 Proof of Proposition 7.2

We assume $f \neq 0$, and we shall deduce a contradiction. We set $\beta_0 := \min \mathcal{S} > 0$. Let L_0 be the minimum of the non-empty set $\{i/e + (j+k)\beta_0 \mid f_{i,j,k} \neq 0\}$. We set $T(L_0) := \{(i, j, k) \in \mathbb{Z}_{\geq 0}^3 \mid f_{i,j,k} \neq 0, i/e + (j+k)\beta_0 = L_0\}$. We set $\ell_0 := \max\{j+k \mid (i, j, k) \in T(L_0)\}$ and $i_0 := e(L_0 - \ell_0\beta_0)$.

Lemma 7.18 *We have $A_{\infty}^{(\ell_0)}(L_0 - \ell_0\beta_0, (\ell_1, \ell_2)) = \ell_1!\ell_2!f_{i_0, \ell_1, \ell_2}$ for any (ℓ_1, ℓ_2) with $\ell_1 + \ell_2 = \ell_0$.*

Proof By definition, we have the following:

$$A_{\infty}^{(\ell_0)}(L_0 - \ell_0\beta_0, (\ell_1, \ell_2)) = \ell_1!\ell_2!f_{i_0, \ell_1, \ell_2} + \sum_{(i, \zeta) \in T_+(\mathcal{S}, L_0 - \ell_0\beta_0)} \sum_{j+k=|\zeta|+\ell_0} f_{i,j,k}(\theta) \frac{j!k!}{(j-\ell_1)!(k-\ell_2)!} \prod_{q=1}^{j-\ell_1} R_{\infty}(\zeta_q) \prod_{q=j-\ell_1+1}^{|\zeta|} I_{\infty}(\zeta_q). \quad (31)$$

Let us observe that the second term in the right hand side of (31) does not appear. Suppose that the summand for $(i, \zeta_1, \dots, \zeta_M, j, k)$ is non-trivial, and we shall derive a contradiction. We have $M > 0$ and $f_{i,j,k} \neq 0$. We have the relations $i/e + \sum_{q=1}^M \zeta_q + \ell_0\beta_0 = L_0$ and $j+k = M + \ell_0$. Because $\beta_0 \leq \zeta_q$, we have

$$i/e + (j+k)\beta_0 = i/e + M\beta_0 + \ell_0\beta_0 \leq i/e + \sum_{q=1}^M \zeta_q + \ell_0\beta_0 = L_0. \quad (32)$$

By our choice of L_0 , the inequality in (32) should be an equality, and we have $\zeta_1 = \dots = \zeta_q = \beta_0$. By our choice of ℓ_0 , we should have $M = 0$, which contradicts with $M > 0$. Hence, we do not have the second term in the right hand side of (31). \blacksquare

We have $A_{\infty}^{(m)}(L, (a, b)) = 0$ for any L and $a+b=m$ by the assumption and Lemma 7.16. In particular, we have $A_{\infty}^{(\ell_0)}(L_0 - \ell_0\beta_0, (\ell_1, \ell_2)) = 0$ for any (ℓ_1, ℓ_2) such that $\ell_1 + \ell_2 = \ell_0$, i.e., $f_{i_0, \ell_1, \ell_2} = 0$ for any (ℓ_1, ℓ_2) such that $\ell_1 + \ell_2 = \ell_0$. But, it contradicts with our choice of ℓ_0 . Hence, we obtain Proposition 7.2. \blacksquare

7.2 Pull back of ramified analytic functions

7.2.1 Setting

Let m_k ($k = 1, 2, \dots$) be an increasing sequence of integers such that $m_k \rightarrow \infty$. Let $\kappa_k \in \frac{1}{m_k}\mathbb{Z}_{>0}$ be an increasing sequence. We set $\hat{\kappa} := \lim_{k \rightarrow \infty} \kappa_k \in \mathbb{R}_{\geq 0} \cup \{\infty\}$. We assume $\hat{\kappa} < \infty$, which implies either one of the following.

Case 1 There exists a prime \mathfrak{p}_0 such that $-\text{ord}_{\mathfrak{p}_0}(\kappa_i) \rightarrow \infty$.

Case 2 We have a sequence of primes \mathfrak{p}_i ($i = 1, 2, \dots$) with $\lim_{i \rightarrow \infty} \mathfrak{p}_i = \infty$ such that $\text{ord}_{\mathfrak{p}_i}(\kappa_i) < 0$ and $\text{ord}_{\mathfrak{p}_i}(\kappa_j) \geq 0$ ($\forall j < i$).

Let $\mathcal{P}^{(k)}(t, \mathfrak{a}) = \sum_{\eta > 0} \mathcal{P}_\eta^{(k)}(\mathfrak{a}) t^\eta \in \mathbb{C}[[t^{1/m_k}, \mathfrak{a}]]$ ($k = 1, 2, \dots$) be a family of elements as follows.

- $\mathcal{P}^{(k)}(t, \mathfrak{a})$ are convergent power series of $(t^{1/m_k}, \mathfrak{a})$.
- $\mathcal{P}_{\kappa_k}^{(k)}(0) \neq 0$.
- $\mathcal{P}_\eta^{(k)}(\mathfrak{a})$ are independent of \mathfrak{a} if $\eta < \kappa_k$. We denote $\mathcal{P}_\eta^{(k)}(\mathfrak{a})$ just by $\mathcal{P}_\eta^{(k)}$ for such η .
- $\mathcal{P}_\eta^{(k)} = \mathcal{P}_\eta^{(k')}$ for $\eta < \min\{\kappa_k, \kappa_{k'}\}$. We also have $\mathcal{P}_{\kappa_k}^{(k)}(0) = \mathcal{P}_{\kappa_k}^{(k')}$ if $k < k'$.
- For any $\eta < \kappa_k$ such that $\mathcal{P}_\eta^{(k)} \neq 0$, we have $\text{ord}_{\mathfrak{p}_0}(\eta) > \text{ord}_{\mathfrak{p}_0}(\kappa_k)$ in **(Case 1)** or $\text{ord}_{\mathfrak{p}_k}(\eta) \geq 0$ in **(Case 2)**.

For any $\eta < \widehat{\kappa}$, we put $\mathcal{P}_\eta := \lim_{k \rightarrow \infty} \mathcal{P}_\eta^{(k)}$. Indeed, we have $\mathcal{P}_\eta = \mathcal{P}_\eta^{(k)}$ if $\eta \leq \kappa_k$. We set

$$\mathcal{S} := \bigcup_k \{0 < \eta < \widehat{\kappa} \mid \mathcal{P}_\eta^{(k)} \neq 0\}.$$

We have $\{0 < \eta < \widehat{\kappa} \mid \mathcal{P}_\eta \neq 0\} \subset \mathcal{S}$. Note that \mathcal{S} is discrete in $\{0 \leq \eta < \widehat{\kappa}\}$.

Let $\phi_1^{(k)} > 0$ ($k = 1, 2, \dots$) be a decreasing sequence. We set $\mathcal{I}^{(k)} := \{-\phi_1^{(k)} < \phi < \phi_1^{(k)}\}$. Let $r_k > 0$ be a decreasing sequence of real numbers, and let $\mathcal{U}^{(k)}$ be a decreasing sequence of neighbourhoods of 0 in $\mathbb{C}_\mathfrak{a}$ such that $\mathcal{P}^{(k)}(t, \mathfrak{a})$ are absolutely convergent on $(t, \mathfrak{a}) \in [0, r_k[\times \mathcal{U}^{(k)}$. We set $J^{(k)} := [0, r_k[\times \mathcal{I}^{(k)} \times \mathcal{U}^{(k)} = \{(t, \phi, \mathfrak{a})\}$. We consider the maps $\varphi^{(k)} : J^{(k)} \longrightarrow \mathbb{R}_{\geq 0} \times \mathbb{R} \times \mathbb{R}^2$ given by

$$\varphi^{(k)}(t, \phi, \mathfrak{a}) = \left(t, \phi, \text{Re}(\mathcal{P}^{(k)}(te^{\sqrt{-1}\phi}, \mathfrak{a})), \text{Im}(\mathcal{P}^{(k)}(te^{\sqrt{-1}\phi}, \mathfrak{a})) \right).$$

7.2.2 Pull back of ramified analytic functions

Let f be a ramified real analytic function on a neighbourhood Y of $(0, 0, 0, 0)$ in $\mathbb{R}_{\geq 0} \times \mathbb{R} \times \mathbb{R}^2$ given by the following power series, where e is a positive integer:

$$f = \sum_{i, j, k \geq 0} f_{i, j, k}(\theta) r^{i/e} x^j y^k.$$

We assume that $f_{0,0,0}(\theta) = 0$ for any θ , and that f is not constantly 0.

We set $f^{(k)}(\phi, t, \mathfrak{a}) := (\varphi^{(k)})^*(f)(\phi, t, \mathfrak{a})$. We have the expansion $f^{(k)}(\phi, t, \mathfrak{a}) = \sum f_L^{(k)}(\phi, \mathfrak{a}) t^L$. We may regard $f_L^{(k)}(\phi, \mathfrak{a})$ as germs of real analytic functions at $(0, 0) \in \mathbb{R} \times \mathbb{C}$.

For any $\eta \in \mathcal{S}$, we set $R(\eta) := \text{Re}(\mathcal{P}_\eta e^{\sqrt{-1}\eta\phi})$ and $I(\eta) := \text{Im}(\mathcal{P}_\eta e^{\sqrt{-1}\eta\phi})$. For any $L \geq 0$, we have the following real analytic functions of $\phi \in \mathbb{R}$:

$$A(L) := \sum_{(i, \zeta) \in T_+(\mathcal{S}, L)} \sum_{\substack{(c_1, c_2) \in \mathbb{Z}_{\geq 0}^2 \\ c_1 + c_2 = |\zeta|}} f_{i, c_1, c_2}(\phi) \prod_{q=1}^{c_1} R(\zeta_q) \prod_{q=c_1+1}^{|\zeta|} I(\zeta_q) + \sum_{i \in T_0(\mathcal{S}, L)} f_{i, 0, 0}(\phi).$$

We may naturally regard $R(\eta)$, $I(\eta)$ and $A(L)$ as germs of real analytic functions at $(0, 0) \in \mathbb{R} \times \mathbb{C}$.

Lemma 7.19 *We have L_1 such that $A(L) = 0$ for any $L < L_1$ and $A(L_1) \neq 0$ as a germ of real analytic functions at $(0, 0) \in \mathbb{R} \times \mathbb{C}$,*

Proof Because f is assumed to be non-constant, the claim follows from Proposition 7.2. ■

Proposition 7.20 *Let L_1 be as in Lemma 7.19. We have k_1 such that the following holds for any $k \geq k_1$:*

- We have $f_L^{(k)} = 0$ for any $L < L_1$ as germs of real analytic functions.
- We have $f_{L_1}^{(k)} = A(L_1)$ as a germ of real analytic functions. In particular, $f_{L_1}^{(k)}$ is independent of k .

Proof For any subset $S \subset \mathbb{R}_{\geq 0}$, we put $T_+(S) := \coprod_{m \geq 1} \mathbb{Z}_{\geq 0} \times S^m$ and $T_0(S) := \mathbb{Z}_{\geq 0}$. For any $L \geq 0$, we set

$$T_+(S, L) := \left\{ (i, s_1, \dots, s_m) \in T_0(S) \mid \frac{i}{e} + \sum_{j=1}^m s_j = L \right\},$$

and $T_0(S, L) := \{i \in T_0(S) \mid i/e = L\}$. We define $T(S) = T_0(S) \sqcup T_+(S)$ and $T(S, L) = T_0(S, L) \sqcup T_+(S, L)$. For any element $\mathbf{s} = (s_1, \dots, s_m) \in S^m$, the number m is denoted by $|\mathbf{s}|$.

We set $\mathcal{S}(\mathcal{P}^{(k)}) := \{\eta \in \mathbb{R} \mid \mathcal{P}_\eta^{(k)} \neq 0\}$. For any $\eta \in \mathcal{S}(\mathcal{P}^{(k)})$, we set $R^{(k)}(\eta) := \text{Re}(\mathcal{P}_\eta^{(k)}(\mathbf{a})e^{\sqrt{-1}\eta\phi})$ and $I^{(k)}(\eta) := \text{Im}(\mathcal{P}_\eta^{(k)}(\mathbf{a})e^{\sqrt{-1}\eta\phi})$. We have the following description:

$$f_L^{(k)} = \sum_{(i, \zeta) \in T_+(\mathcal{S}(\mathcal{P}^{(k)}), L)} \sum_{\substack{(c_1, c_2) \in \mathbb{Z}_{\geq 0}^2 \\ c_1 + c_2 = |\zeta|}} f_{k, c_1, c_2} \prod_{q=1}^{c_1} R^{(k)}(\zeta_q) \prod_{q=c_1+1}^{|\zeta|} I^{(k)}(\zeta_q) + \sum_{i \in T_0(\mathcal{S}(\mathcal{P}^{(k)}), L)} f_{i, 0, 0}.$$

We set $\mathcal{S}_{<\hat{\kappa}}(\mathcal{P}^{(k)}) := \{\eta \in \mathbb{R} \mid \mathcal{P}_\eta^{(k)} \neq 0, \eta < \hat{\kappa}\}$. For $(m, \ell_1, \ell_2) \in \mathbb{Z}_{\geq 0}^3$ such that $m = \ell_1 + \ell_2$, we put

$$\begin{aligned} A_{<\hat{\kappa}}^{(m)}(\mathcal{P}^{(k)}, L, (\ell_1, \ell_2)) := & \sum_{(i, \zeta) \in T_+(\mathcal{S}_{<\hat{\kappa}}(\mathcal{P}^{(k)}), L)} \sum_{\substack{(c_1, c_2) \in \mathbb{Z}_{\geq 0}^2 \\ c_1 + c_2 = |\zeta| + m}} f_{k, c_1, c_2}(\phi) \frac{c_1! c_2!}{(c_1 - \ell_1)!(c_2 - \ell_2)!} \prod_{q=1}^{c_1 - \ell_1} R^{(k)}(\zeta_q) \prod_{q=c_1 - \ell_1 + 1}^{|\zeta|} I^{(k)}(\zeta_q) \\ & + \sum_{i \in T_0(\mathcal{S}_{<\hat{\kappa}}(\mathcal{P}^{(k)}), L)} \ell_1! \ell_2! f_{i, \ell_1, \ell_2}(\phi). \end{aligned} \quad (33)$$

For any $\delta \geq 0$ and $(b_1, b_2) \in \mathbb{Z}_{\geq 0}^2$, we set

$$\mathcal{U}^{(k)}(\delta, b_1, b_2) := \left\{ (\zeta_1, \dots, \zeta_{b_1+b_2}) \in \mathcal{S}(\mathcal{P}^{(k)})^{b_1+b_2} \mid \zeta_i \geq \hat{\kappa}, \sum \zeta_i = \delta \right\}.$$

We also define

$$B^{(k)}(\delta, b_1, b_2) := \sum_{\zeta \in \mathcal{U}^{(k)}(\delta, b_1, b_2)} \frac{1}{b_1! b_2!} \prod_{q=1}^{b_1} R^{(k)}(\zeta_q) \prod_{q=b_1+1}^{b_1+b_2} I^{(k)}(\zeta_q).$$

Note that for each k the set $\{\delta \mid \mathcal{U}^{(k)}(\delta, b_1, b_2) \neq \emptyset\}$ is discrete. For any $L \leq L_1$, we have the following equality:

$$f_L^{(k)} = \sum_{d \geq 0} \sum_{\delta \geq 0} \sum_{\ell_1 + \ell_2 = d} A_{<\hat{\kappa}}^{(d)}(\mathcal{P}^{(k)}, L - \delta, (\ell_1, \ell_2)) \cdot B^{(k)}(\delta, \ell_1, \ell_2).$$

By Proposition 7.1, we have j_0 such that $A_{<\hat{\kappa}}^{(d)}(\mathcal{P}^{(k)}, L - \sum \zeta_j, (\ell_1, \ell_2)) = 0$ for any $L \leq L_1$, any $k \geq k_0$ and any $(\ell_1, \ell_2) \in \mathbb{Z}_{\geq 0}^2$ satisfying $\ell_1 + \ell_2 = d \geq 1$. We obtain $f_L^{(k)} = 0$ for any $L < L_1$ and any $k \geq k_0$. We also obtain $f_{L_1}^{(k)} = A(L_1) \neq 0$ for any $k \geq k_0$. Hence, we obtain Proposition 7.20. ■

Corollary 7.21 *We have a discrete subset $Z \subset \mathbb{R}$ and $k_0 \in \mathbb{Z}_{>0}$ such that the following holds for any $k \geq k_0$:*

- The orders $\text{ord}_t f^{(k)}(\phi, t, \mathbf{a})$ for $(\phi, \mathbf{a}) \in (\mathcal{I}^{(k)} \setminus Z) \times \mathcal{U}^{(k)}$ are constant and independent of k . Note that the functions $f^{(k)}(\phi, t, \mathbf{a})$ are not constant. ■

7.3 Formal paths

7.3.1 Ringed spaces

Let Y be a real analytic manifold. Let $\mathcal{O}_Y^{\mathbb{R}}$ denote the sheaf of real analytic functions on Y . For any open subset U of Y , let $\mathcal{N}'_Y(U)$ be the Novikov type ring over $\mathcal{O}_Y^{\mathbb{R}}(U)$, i.e.,

$$\mathcal{N}'_Y(U) := \left\{ \sum_{i=0}^{\infty} b_i t^{\eta_i} \mid b_i \in \mathcal{O}_Y^{\mathbb{R}}(U), \eta_i \in \mathbb{Q}_{\geq 0}, \eta_i < \eta_{i+1}, \lim_{i \rightarrow \infty} \eta_i = \infty \right\}.$$

The correspondence $U \mapsto \mathcal{N}'_Y(U)$ gives a presheaf on Y . Let \mathcal{N}_Y denote the associated sheaf. We have $\mathcal{N}_Y(U) = \mathcal{N}'_Y(U)$ if U is connected. Thus, we obtain a sheaf of algebras \mathcal{N}_Y on Y . We formally denote sections s of \mathcal{N}_Y on U as $\sum a_{\eta} t^{\eta}$. We have the natural morphism of pre-sheaves $\mathcal{N}'_Y \rightarrow \mathcal{O}_Y^{\mathbb{R}}$ given by $\sum_{\eta} a_{\eta} t^{\eta} \mapsto a_0$. It induces a morphism of sheaves $\mathcal{N}_Y \rightarrow \mathcal{O}_Y^{\mathbb{R}}$.

For any element $s = \sum a_{\eta} t^{\eta}$ in the stalk $\mathcal{N}_{Y,P}$, we put $\text{ord}_{P,t}(s) := \min\{\eta \mid a_{\eta} \neq 0\}$ if $s \neq 0$, and $\text{ord}_{P,t}(s) := \infty$ if $s = 0$. Here, $a_{\eta} \neq 0$ means that a_{η} is not constantly 0 on a neighbourhood of P .

Let P be any point of Y . Let $\iota_P : \{P\} \rightarrow Y$ denote the inclusion. We have the natural morphism $\iota_P^{-1} \mathcal{N}_Y \rightarrow \mathcal{N}_P$. For any section s of \mathcal{N}_Y on an open set $U \subset Y$, and for any $P \in U$, let s_P and $s|_P$ denote the induced elements of $\mathcal{N}_{Y,P}$ and \mathcal{N}_P , respectively. We have $\text{ord}_{P,t}(s) \leq \text{ord}_{P,t}(s|_P)$.

Let s be an element of $\mathcal{N}_{Y,P}$. We have a small neighbourhood Y_P of P such that $s = \sum a_{\eta} t^{\eta}$, where $a_{\eta} \in \mathcal{O}_Y^{\mathbb{R}}(Y_P)$. We say that s is convergent if the following holds.

- We have $m \in \mathbb{Z}_{>0}$ such that $a_{\eta} = 0$ unless $m\eta \in \mathbb{Z}$. Moreover, s comes from an analytic function on $Y_P \times I_{t^{1/m}}$.

7.3.2 Morphisms of ringed spaces

Let M be any real analytic manifold. An \mathcal{N}_Y -path in M is a morphism of ringed spaces $\varphi : (Y, \mathcal{N}_Y) \rightarrow (M, \mathcal{O}_M^{\mathbb{R}})$, i.e., it consists of a real analytic map $\varphi_0 : Y \rightarrow M$ and a homomorphism of sheaves of algebras $\varphi_0^{-1} \mathcal{O}_M^{\mathbb{R}} \rightarrow \mathcal{N}_Y$ such that the composite with $\mathcal{N}_Y \rightarrow \mathcal{O}_Y^{\mathbb{R}}$ is equal to the natural map $\varphi_0^* : \mathcal{O}_M^{\mathbb{R}} \rightarrow \mathcal{O}_Y^{\mathbb{R}}$.

Let $\iota : Y_1 \subset Y$ be any real analytic submanifold. For any \mathcal{N}_Y -path φ , we have the naturally induced \mathcal{N}_{Y_1} -path $\varphi|_{Y_1} := \varphi \circ \iota$.

Let φ, ψ be \mathcal{N}_Y -paths in M . Let $P \in Y$. Suppose that $\varphi_0(P) = \psi_0(P)$. Then, we set $\text{ord}_{P,t}(\varphi, \psi) = \min_i \text{ord}_t(\varphi^*(x_i) - \psi^*(x_i))$, where (x_1, \dots, x_n) be a real analytic coordinate around $\varphi_0(P)$ such that $x_i(\varphi_0(P)) = 0$. It is easy to see that the number $\text{ord}_{P,t}(\varphi, \psi)$ is independent of the choice of (x_1, \dots, x_n) . If $\varphi_0(P) \neq \psi_0(P)$, then we set $\text{ord}_{P,t}(\varphi, \psi) = 0$. Let $Y_1 \subset Y$ be any real analytic manifold such that $P \in Y_1$. We have $\text{ord}_{P,t}(\varphi, \psi) \leq \text{ord}_{P,t}(\varphi|_{Y_1}, \psi|_{Y_1})$.

Suppose that M is equipped with a global coordinate (x_1, \dots, x_n) . We also assume that Y is connected. Then, \mathcal{N}_Y -path in M is equivalent to a tuple $(f_1, \dots, f_n) \in \mathcal{N}_Y(Y)^n$. The correspondence is given by $\varphi \mapsto f_i = \varphi^*(x_i)$ ($i = 1, \dots, n$).

7.3.3 Real blowing up

Let C be a real analytic submanifold in M . We say that an \mathcal{N}_Y -path φ factors through C if it factors through $(C, \mathcal{O}_C^{\mathbb{R}})$ as a morphism of ringed spaces.

Suppose that an \mathcal{N}_Y -path φ does not factor through C . Let $P \in Y$. We take a real analytic coordinate (x_1, \dots, x_n) around $\varphi_0(P)$ such that $C = \{x_1 = \dots = x_{\ell} = 0\}$. If $\varphi_0(P) \in C$, we set

$$\text{ord}_{P,t}(\varphi, C) := \min_{1 \leq i \leq \ell} \text{ord}_{P,t} \varphi^*(x_i).$$

It is easy to see that $\text{ord}_{P,t}(\varphi, C)$ is independent of the choice of the coordinate (x_1, \dots, x_n) . Note that for any $P \in Y$ we have the induced \mathcal{N}_P -path $\varphi|_P$ of M , for which $\text{ord}_{P,t}(\varphi|_P, C) \geq \text{ord}_{P,t}(\varphi, C)$.

Lemma 7.22 *Let φ be an \mathcal{N}_Y -path in M which does not factor through C . Let P be a point of M . Suppose $\text{ord}_{P,t}(\varphi, C) = \text{ord}_{P,t}(\varphi|_P, C)$. Let $\text{Bl}_C M$ denote the real blowing up of M along C .*

- We have a neighbourhood Y_1 of P in Y and a \mathcal{N}_{Y_1} -path $\tilde{\varphi}$ in $\text{Bl}_C M$ such that $p \circ \tilde{\varphi} = \varphi|_{Y_1}$, where $p : \text{Bl}_C(M) \rightarrow M$ denotes the projection.
- If we have another neighbourhood Y_2 of P and \mathcal{N}_{Y_2} -path $\tilde{\varphi}'$ in $\text{Bl}_C M$ such that $p \circ \tilde{\varphi}' = \varphi|_{Y_2}$. Then, we have $\tilde{\varphi}|_{Y_1 \cap Y_2} = \tilde{\varphi}'|_{Y_1 \cap Y_2}$.

Proof By shrinking Y and M , we may assume that M is equipped with a coordinate (x_1, \dots, x_n) such that $C = \{x_1 = \dots = x_\ell = 0\}$. The \mathcal{N}_Y -path φ is expressed as $(\varphi_1, \dots, \varphi_n) \in \mathcal{N}_Y(Y)^n$. We have the expressions $\varphi_i = \sum_{\eta \in \mathbb{R}} \varphi_{i,\eta} t^\eta$. We may assume that $\text{ord}_{Q,t}(\varphi, C) = \text{ord}_{Q,t}(\varphi_1) =: \eta_0$ for any Q , and that φ_{1,η_0} is nowhere vanishing.

We have $\text{Bl}_C(M) = \{(x_1, \dots, x_n), [y_1 : \dots : y_\ell] \mid x_i y_j - x_j y_i = 0, (1 \leq i, j \leq \ell)\}$. We have the open subset $U_k \subset \text{Bl}_C(M)$ ($k = 1, \dots, \ell$) determined by the condition $y_k \neq 0$. We have the coordinate $(u_1^{(k)}, \dots, u_n^{(k)})$ of U_k given by $u_j^{(k)} = x_j/x_k$ if $1 \leq j \leq \ell$ and $j \neq k$, by $u_j^{(k)} = x_j$ if $j = k$ or $j > \ell$. We have the \mathcal{N}_Y -path $\tilde{\varphi}$ in U_1 given as follows with respect to $(u_1^{(1)}, \dots, u_n^{(1)})$:

$$(\varphi_1, \varphi_2/\varphi_1, \dots, \varphi_\ell/\varphi_1, \varphi_{\ell+1}, \dots, \varphi_n).$$

We clearly have $p \circ \tilde{\varphi} = \varphi$.

Suppose that we have an \mathcal{N}_Y -path $\tilde{\varphi}'$ in U_k such that $p \circ \tilde{\varphi}' = \varphi$. It is expressed as $(\tilde{\varphi}'_1, \dots, \tilde{\varphi}'_n)$ with respect to $(u_1^{(k)}, \dots, u_n^{(k)})$. We have $\varphi_1 = \tilde{\varphi}'_1 \cdot \tilde{\varphi}'_k$ and $\varphi_k = \tilde{\varphi}'_k$. By comparing the orders of φ_1 and φ_k , we obtain that $\tilde{\varphi}'_1$ is invertible. Then, it is easy to see that $\tilde{\varphi}'$ is equal to $\tilde{\varphi}$ as an \mathcal{N}_Y -path in $\text{Bl}_C(M)$. \blacksquare

Let M , φ and $P \in Y$ be as in Lemma 7.22. Let φ' be another \mathcal{N}_Y -path in M with the following property.

- The underlying map $\varphi_0, \varphi'_0 : Y \rightarrow M$ are the same on a neighbourhood of P .
- $\text{ord}_{P,t}(\varphi', \varphi) > \text{ord}_{P,t}(\varphi, C)$.

Then, we can easily check the following by a direct computation.

Lemma 7.23 φ' does not factor through C , and $\text{ord}_{P,t}(\varphi', C) = \text{ord}_{P,t}(\varphi'_P, C)$. \blacksquare

If Y_1 is a small neighbourhood of P in Y , then we have \mathcal{N}_{Y_1} -paths $\tilde{\varphi}$ and $\tilde{\varphi}'$ in $\text{Bl}_C M$ such that $p \circ \tilde{\varphi} = \varphi|_{Y_1}$ and $p \circ \tilde{\varphi}' = \varphi'|_{Y_1}$. We can check the following by a direct computation.

Lemma 7.24 The underlying analytic maps $\tilde{\varphi}_0, \tilde{\varphi}'_0 : Y_1 \rightarrow M$ are the same. Moreover, we have

$$\text{ord}_{Q,t}(\tilde{\varphi}, \tilde{\varphi}') = \text{ord}_{Q,t}(\varphi, \varphi') - \text{ord}_{Q,t}(\varphi, C)$$

at any $Q \in Y_1$. \blacksquare

7.3.4 A complement on convergence

Set $I := \{0 \leq \theta \leq 1\}$ and $J := \{0 \leq r \leq 1\}$. Suppose that we are given real analytic maps $\iota : M \rightarrow X$ and $p : X \rightarrow I \times J$. We set $F := p \circ \iota$. We assume that (i) $\dim M = 2$, (ii) M is connected, (iii) $I \times \{0\} \subsetneq F(M)$, (iv) $\dim F^{-1}(I \times \{0\}) = 1$.

We have the \mathcal{N}_I -path γ_0 in $I \times J$ given by the identity map $I \simeq I \times \{0\}$ and the correspondence $r \mapsto t$.

Lemma 7.25 Let φ be an \mathcal{N}_I -path in X which factors through M . We assume the following.

- $p \circ \varphi$ is the \mathcal{N}_I -path γ_0 .

Then, we have a 0-dimensional subset $Z \subset I$ such that φ is convergent at any $P \in I \setminus Z$ in the following sense.

- We take a coordinate neighbourhood (X_P, x_1, \dots, x_n) of X around $\varphi_0(P)$. We have the expression $(\varphi_1, \dots, \varphi_n)$ of φ with respect to the coordinate. Then, φ_i are convergent.

Proof We have a 0-dimensional analytic subset $Z_0 \subset F^{-1}(I \times \{0\})$ such that the following holds.

- $F^{-1}(I \times \{0\}) \setminus Z_0$ is a smooth submanifold in M_1 .
- Take any point $P_1 \in F^{-1}(I \times \{0\}) \setminus Z_0$. Take a real analytic coordinate $(U; x, y)$ around P_1 such that $x^{-1}(0) = U \cap F^{-1}(I \times \{0\})$. Then, $U \rightarrow J$ is expressed as $\sum_{j=e}^{\infty} a_j(y)x^j$ such that a_e is nowhere vanishing on $\{x = 0\}$.

Then, the claim of the lemma is clear. ■

8 Real local blowings up and complex blowings up

8.1 Statements

8.1.1 Infinite sequence of complex blowings up

We use the notation in §5.3. Let $\mathbf{Y} = (\eta_1, \omega_1, \eta_2, \omega_2, \dots) \in \mathfrak{P}^\infty$, where $\eta_i \in \{+, -\}^{\ell(i)}$ and $\omega_i \in \mathbb{C}^*$. We assume the following.

- The limit curve for \mathbf{Y} is not convergent.

We put $\mathbf{Y}_m := (\eta_1, \omega_1, \dots, \eta_m, \omega_m)$. We set $X_m := \text{Bl}_{\mathbf{Y}_m} \mathbb{C}^2$. We have the sequence of morphisms of complex manifolds

$$\cdots \rightarrow X_{m+1} \rightarrow X_m \rightarrow X_{m-1} \rightarrow \cdots \rightarrow X_0 = \mathbb{C}^2.$$

Recall that we have the points $P_m := P_{\mathbf{Y}_m} \in X_m$ and the coordinate neighbourhood $(U_m, u_m, v_m) := (U_{\mathbf{Y}_m}, u_{\mathbf{Y}_m}, v_{\mathbf{Y}_m})$ around P_m . For $m' \geq m$, the induced maps $\psi_{\mathbf{Y}_m, \mathbf{Y}_{m'}} : \text{Bl}_{\mathbf{Y}_{m'}} \mathbb{C}^2 \rightarrow \text{Bl}_{\mathbf{Y}_m} \mathbb{C}^2$ are denoted by $\psi_{m, m'} : X_{m'} \rightarrow X_m$. The induced morphisms $\psi_{\mathbf{Y}_m} : \text{Bl}_{\mathbf{Y}_m} \mathbb{C}^2 \rightarrow \mathbb{C}^2$ are denoted by $\psi_m : X_m \rightarrow X_0$.

Let $H_0 := \{y = 0\} \subset X_0$. We set $H_m := \psi_m^{-1}(H_0)$. Let $\varpi_m : \tilde{X}_m(H_m) \rightarrow X_m$ be the oriented real blowing up along H_m . We have the induced morphisms $\tilde{\psi}_m : \tilde{X}_m(H_m) \rightarrow \tilde{X}_0(H_0)$ and $\tilde{\psi}_{m, m'} : \tilde{X}_{m'}(H_{m'}) \rightarrow \tilde{X}_m(H_m)$.

Note that P_m are smooth points of normal crossing hypersurfaces H_m . Let \mathcal{U}_m denote a small neighbourhood of 0 in $\{u_m \in \mathbb{C}\}$. Let $\Delta_{m, \epsilon} := \{|v_m| < \epsilon\}$. We can naturally regard $\mathcal{U}_m \times \Delta_{m, \epsilon}$ as a neighbourhood of P_m in U_m . Let $\tilde{\Delta}_{m, \epsilon}(0) \rightarrow \Delta_{m, \epsilon}$ denote the oriented real blowing up at 0. We can naturally regard $\mathcal{U}_m \times \tilde{\Delta}_{m, \epsilon}(0)$ as a neighbourhood of $\varpi_m^{-1}(P_m)$ in $\tilde{X}_m(H_m)$. We can naturally identify $\tilde{\Delta}_{m, \epsilon}(0)$ with $\partial \tilde{\Delta}_{m, \epsilon}(0) \times [0, \epsilon[$, and hence $\mathcal{U}_m \times \tilde{\Delta}_{m, \epsilon}(0) \simeq \mathcal{U}_m \times \partial \tilde{\Delta}_{m, \epsilon}(0) \times [0, \epsilon[$. If we are given an open subset $\mathcal{I} \subset \varpi_m^{-1}(P_m)$ and a positive number $\epsilon' > 0$, then $\mathcal{I} \times]0, \epsilon'[$ and $\mathcal{I} \times [0, \epsilon'[$ are naturally regarded as subsets in $\tilde{X}_m(H_m)$.

8.1.2 Lifting with respect to composition of local real blowings up

We identify $\tilde{X}_0(H_0) = \mathbb{C}_x \times \tilde{\mathbb{C}}_y(0)$ with $\mathcal{Y}_+ := \mathbb{C} \times \mathbb{R}_{\geq 0} \times S^1$. We naturally embed it into $\mathcal{Y} := \mathbb{C} \times \mathbb{R} \times S^1$. In this way, we regard $\tilde{X}_0(H_0)$ as a closed subset in \mathcal{Y} .

Let $\phi : W \rightarrow \mathcal{Y}$ be a morphism obtained as the composite of local real blowings up:

$$W = W^{(k)} \xrightarrow{\phi^{(k)}} W^{(k-1)} \xrightarrow{\phi^{(k-1)}} \cdots \xrightarrow{\phi^{(2)}} W^{(1)} \xrightarrow{\phi^{(1)}} W^{(0)} = \mathcal{Y}.$$

Moreover, we have subanalytic open subsets $U^{(p)} \subset W^{(p)}$ and closed real analytic submanifolds $C^{(p)}$, and the morphisms $\phi^{(p+1)} : W^{(p+1)} \rightarrow W^{(p)}$ are obtained as the real blowing up of $U^{(p)}$ along $C^{(p)}$. We may assume that $C^{(p)}$ are the complete intersection of the real analytic functions $g_1^{(p)}, \dots, g_{r(p)}^{(p)}$ on $U^{(p)}$ such that $dg_1^{(p)}, \dots, dg_{r(p)}^{(p)}$ are linearly independent at each point of $C^{(p)}$. Let $\xi^{(p)} : W^{(p)} \rightarrow \mathcal{Y}$ denote the induced map.

Suppose that we are given a sequence of numbers $m_i \rightarrow \infty$ a sequence of points $Q_{m_i} \in \varpi_{m_i}^{-1}(P_{m_i})$, and a sequence of neighbourhoods \mathcal{I}_{m_i} of Q_{m_i} in $\varpi_{m_i}^{-1}(P_{m_i})$, such that the following holds.

E1 We have $\tilde{\psi}_{m_i, m_j}(Q_{m_j}) = Q_{m_i}$ for $m_j > m_i$.

E2 For small positive numbers r_{m_i} , the images $\widetilde{\psi}_{m_i}(\mathcal{I}_{m_i} \times]0, r_{m_i}[)$ do not intersect with the set of the critical values of ϕ . Moreover, we have analytic maps $\widetilde{\psi}_{m_i,0} : \mathcal{I}_{m_i} \times [0, r_{m_i}[\rightarrow W$ such that $\phi \circ \widetilde{\psi}_{m_i,0}$ are equal to the restriction of $\widetilde{\psi}_{m_i}$ to $\mathcal{I}_{m_i} \times [0, r_{m_i}[$. Note that such $\widetilde{\psi}_{m_i,0}$ are uniquely determined by the property.

We shall prove the following proposition later (§8.2.1 and §8.3.3).

Proposition 8.1 *We have i_1 such that the following holds for each $i \geq i_1$:*

- We have a non-empty open subset $\mathcal{I}'_{m_i} \subset \mathcal{I}_{m_i}$, a neighbourhood \mathcal{V}_{m_i} of \mathcal{I}'_{m_i} in $\widetilde{X}_{m_i}(H_{m_i})$ and a real analytic map

$$\widehat{\psi}_{m_i} : \mathcal{V}_{m_i} \rightarrow W$$

such that $\phi \circ \widehat{\psi}_{m_i}$ is equal to the restriction of $\widetilde{\psi}_{m_i}$ to \mathcal{V}_{m_i} .

8.1.3 Lifting with respect to covering by the composition of local blowings up

Let $(W_\lambda, \phi_\lambda)$ ($\lambda \in \Lambda$) be a finite family of analytic maps $\phi_\lambda : W_\lambda \rightarrow \mathcal{Y}$ such that (i) ϕ_λ are the composition of local real blowings up, (ii) we have subanalytic compact subsets $K_\lambda \subset W_\lambda$ such that $\bigcup \phi_\lambda(K_\lambda)$ contains the neighbourhood of $\varpi_0^{-1}(0, 0)$.

Proposition 8.2 *We have $\lambda_0 \in \Lambda$, a sequence $m_i \rightarrow \infty$, a sequence of points $Q_{m_i} \in \varpi_{m_i}^{-1}(P_{m_i})$, and a sequence of neighbourhoods \mathcal{I}_{m_i} of Q_{m_i} in $\varpi_{m_i}^{-1}(P_{m_i})$, such that the conditions **E1** and **E2** hold. In particular, we may apply Proposition 8.1 to ϕ_{λ_0} .*

Proof Let $\text{Crit}(\phi_\lambda)$ denote the set of the critical values of ϕ_λ . We have $\dim_{\mathbb{R}} \text{Crit}(\phi_\lambda) \leq 2$. Let $q : \mathcal{Y} = \mathbb{C} \times (S^1 \times \mathbb{R}) \rightarrow S^1 \times \mathbb{R}$ be the projection. By construction, we have the identification $\widetilde{X}_0(H_0) = q^{-1}(S^1 \times \mathbb{R}_{\geq 0})$. It is standard that we have a 0-dimensional subanalytic subset $Z_0 \subset S^1 \times \{0\}$ with the following property.

- For any $P \in (S^1 \times \{0\}) \setminus Z_0$, we have a neighbourhood \mathcal{U}_P of P in $S^1 \times \mathbb{R}_{\geq 0}$ and continuous subanalytic functions f_1^P, \dots, f_ℓ^P on $(\mathcal{U}_P, S^1 \times \mathbb{R})$ to \mathbb{C} such that $\bigcup_\lambda \text{Crit}(\phi_\lambda) \cap q^{-1}(\mathcal{U}_P)$ is the union of the graph of f_p^P .

We regard $Z_0 \subset \varpi_0^{-1}(0, 0)$ by the natural isomorphism $\varpi_0^{-1}(0, 0) \simeq S^1 \times \{0\}$.

Lemma 8.3 *If m_0 is sufficiently large, for each $m \geq m_0$, we have a finite subset $Z_m \subset \varpi_m^{-1}(P_m)$ with the following property.*

- We have $\widetilde{\psi}_m^{-1}(Z_0) \subset Z_m$.
- For any point $P \in \varpi_m^{-1}(P_m) \setminus Z_m$, we have a neighbourhood \mathcal{U}_P of P in $\{0\} \times \widetilde{\Delta}_{m,\epsilon}(0)$ and $\lambda(P) \in \Lambda$ such that we have a unique morphism $\widehat{\psi}_{m,\lambda(P),P} : \mathcal{U}_P \rightarrow W$ such that $\phi_{\lambda(P)} \circ \widehat{\psi}_{m,\lambda(P),P}$ is equal to the restriction of $\widetilde{\psi}_m$ to \mathcal{U}_P .

Proof Note that $\widetilde{\psi}_m(\{0\} \times \widetilde{\Delta}_{m,\epsilon}(0))$ is described as the graph of the multivalued functions $S^1 \times [0, \rho_m[\rightarrow \mathbb{C}$ induced by $g_{Y_m|\{0\} \times \Delta_y}$, where g_{Y_m} are given as in §5.3. The limit curve is assumed to be non-convergent. Hence, if m_1 is sufficiently large, for each $m \geq m_1$, the intersection of $\widetilde{\psi}_m(\{0\} \times \widetilde{\Delta}_{m,\epsilon}(0))$ and $\bigcup \text{Crit}(\phi_\lambda)$ is at most one dimensional.

For $m \geq m_1$, we have the subanalytic subsets $M_{m,\lambda} := \widetilde{\psi}_m^{-1}(\phi_\lambda(K_\lambda)) \cap (\{0\} \times \widetilde{\Delta}_{m,\epsilon}(0))$. We have the following decomposition into connected components:

$$M_{m,\lambda} \setminus \left((\{0\} \times \partial \widetilde{\Delta}_{m,\epsilon}(0)) \cup \widetilde{\psi}_m^{-1} \left(\bigcup \text{Crit}(\phi_\lambda) \right) \right) = \coprod \mathcal{C}_{m,\lambda,j}.$$

We have the unique morphisms $g_{m,\lambda,j} : \mathcal{C}_{m,\lambda,j} \rightarrow W_\lambda$ such that $\phi_\lambda \circ g_{m,\lambda,j}$ is equal to the restriction of $\widetilde{\psi}_m$ to $\mathcal{C}_{m,\lambda,j}$. According to Lemma 2.19, for each (λ, j) such that $\dim_{\mathbb{R}} \mathcal{C}_{m,\lambda,j} = 2$, we have a finite subset $N_{m,\lambda,j} \subset \overline{\mathcal{C}}_{m,\lambda,j}$ and a real analytic map $\overline{g}_{m,\lambda,j} : \overline{\mathcal{C}}_{m,\lambda,j} \setminus N_{m,\lambda,j} \rightarrow W_\lambda$ such that (i) $N_{m,\lambda,j}$ contains the singular locus of $\partial \overline{\mathcal{C}}_{m,\lambda,j}$, (ii) $\phi_\lambda \circ \overline{g}_{m,\lambda,j}$ is equal to the restriction of $\widetilde{\psi}_m$ to $\overline{\mathcal{C}}_{m,\lambda,j} \setminus N_{m,\lambda,j}$.

We have $\tilde{\Delta}_{m,\epsilon}(0) = \bigcup_{\lambda,j} \tilde{\mathcal{C}}_{m,\lambda,j}$. Hence, we have a finite set $Z_m \subset \partial\tilde{\Delta}_{m,\epsilon}(0)$ such that (i) $Z_m \supset \tilde{\psi}_m^{-1}(Z_0)$, (ii) for any $P \in \varpi_m^{-1}(P_m) \setminus Z_m$, we have $(\lambda(P), j(P))$ such that $P \in \tilde{\mathcal{C}}_{m,\lambda(P),j(P)} \setminus N_{m,\lambda(P),j(P)}$. Then, Z_m has the desired property. \blacksquare

The union $Z'_0 := \bigcup_m \tilde{\psi}_m(Z_m)$ is a countable subset in $\varpi_0^{-1}(0,0)$. We have $Q_0 \in \varpi_0^{-1}(0,0) \setminus Z'_0$. We take $Q_m \in \varpi_m^{-1}(P_m)$ such that $\tilde{\psi}_{m,m'}(Q_{m'}) = Q_m$ and that $\tilde{\psi}_m(Q_m) = Q_0$. We can take neighbourhoods \mathcal{I}_m of Q_m in $\varpi_m^{-1}(P_m)$ such that (i) $\tilde{\psi}_m(\mathcal{I}_m) \supset \tilde{\psi}_{m'}(\mathcal{I}_{m'})$ for $m \leq m'$, (ii) $\mathcal{I}_m \cap \bigcup_{j \leq m} \tilde{\nu}_{j,m}^{-1}(Z_j) = \emptyset$. Hence, we have $\lambda_0 \in \Lambda$, a sequence $m_i \rightarrow \infty$, a sequence of points $Q_{m_i} \in \varpi_{m_i}^{-1}(P_{m_i})$, a sequence of neighbourhoods \mathcal{I}_{m_i} of Q_{m_i} in $\varpi_{m_i}^{-1}(P_{m_i})$ such that the conditions **E1** and **E2** are satisfied for $(W_{\lambda_0}, \phi_{\lambda_0})$. \blacksquare

8.1.4 Pull back of subanalytic functions in the case $\hat{\kappa}(\mathbf{Y}) < \infty$

Let Z be a closed subanalytic subset in $\tilde{X}_0(H_0)$ with $\dim_{\mathbb{R}} Z \leq 3$. Let f be a continuous subanalytic function on $(\tilde{X}_0(H_0) \setminus Z, \tilde{X}_0(H_0))$. We assume that f is bounded around any point of $Z \setminus \partial\tilde{X}_0(H_0)$. We shall prove the following proposition in §8.2.2.

Proposition 8.4 *If $\hat{\kappa}(\mathbf{Y}) < \infty$, we have m_0 such that the following holds for each $m \geq m_0$.*

- We have a non-empty connected open subset $\mathcal{I}_m \subset \varpi_m^{-1}(P_m)$ and a small open neighbourhood \mathcal{V}_m of \mathcal{I}_m in $\tilde{X}_m(H_m)$ such that $\tilde{\psi}_m(\mathcal{V}_m \setminus \varpi_m^{-1}(H_m)) \subset \tilde{X}_0(H_0) \setminus Z$.
- The function $\tilde{\psi}_m^*(f)$ on $\mathcal{V}_m \setminus \varpi_m^{-1}(H_m)$ is ramified analytic along $\mathcal{V}_m \cap \varpi_m^{-1}(H_m)$.
- If $\tilde{\psi}_m^*(f)$ is not constantly 0, the order $\text{ord}_{\rho_m} \tilde{\psi}_m^*(f)(u_m, \theta_m, \rho_m)$ are independent of u_m , where (θ_m, ρ_m) is a polar coordinate of $\tilde{\mathbb{C}}_{v_m}(0)$ given by $v_m = \rho_m e^{\sqrt{-1}\theta_m}$.

8.1.5 Pull back of subanalytic functions in the case $\hat{\kappa}(\mathbf{Y}) = \infty$

In this subsection, we assume that \mathbf{Y} is not convergent and that $\hat{\kappa}(\mathbf{Y}) = \infty$. We shall prove the propositions in §8.3.5.

Any small neighbourhood \mathcal{U}_m of 0 in $\mathbb{C}_{u_m} = \{u_m \in \mathbb{C}\}$ naturally induces a subset $\mathcal{U}_m \times \{0\}$ of $U_m = (\mathbb{C}_{u_m} \setminus \{-\omega_m\}) \times \mathbb{C}_{v_m} \subset \tilde{X}_m(H_m)$. Let $u_m = a_m + \sqrt{-1}b_m$ denote the decomposition into the real part and the imaginary part. We consider open sets \mathcal{U}_m of the form $\{(a_m, b_m) \mid |a_m| < \delta_{1,m}, |b_m| < \delta_{2,m}\}$.

Let Z be a closed subanalytic subset in $\tilde{X}_0(H_0)$ with $\dim_{\mathbb{R}} Z = 3$. Let Z_1 be a closed subanalytic subset in Z with $\dim_{\mathbb{R}} Z_1 = 2$.

Proposition 8.5 *There exists m_0 such that for any $m \geq m_0$ we have a non-empty connected open subset $\mathcal{I}_m \subset \varpi_m^{-1}(P_m)$ and a small neighbourhood $\mathcal{V}_m = \mathcal{U}_m \times \mathcal{I}_m \times \{0 \leq \rho_m < \epsilon_m\}$ of \mathcal{I}_m in $\tilde{X}_m(H_m)$ with the following property.*

- Set $\mathcal{Z}_m := (\mathcal{V}_m \cap \tilde{\psi}_m^{-1}(Z)) \setminus \varpi_m^{-1}(H_m)$. Then, we have either (i) \mathcal{Z}_m is empty, or (ii) \mathcal{Z}_m is a smooth connected hypersurface.
- In the case (ii), we have either one of the following isomorphisms:

$$\mathcal{Z}_m \simeq \{|a_m| < \delta_{m,1}\} \times \mathcal{I}_m \times \{0 < \rho_m < \epsilon\} =: \mathcal{V}_{m,1}, \quad (34)$$

$$\text{or } \mathcal{Z}_m \simeq \{|b_m| < \delta_{m,2}\} \times \mathcal{I}_m \times \{0 < \rho_m < \epsilon\} =: \mathcal{V}_{m,2}. \quad (35)$$

Here, the isomorphism is induced by the projection $u_m \mapsto a_m$ or $u_m \mapsto b_m$.

- We have $(\mathcal{V}_m \cap \tilde{\psi}_m^{-1}(Z_1)) \setminus \varpi_m^{-1}(H_m) = \emptyset$.

Let f be a continuous subanalytic function on $(\tilde{X}_0(H_0) \setminus Z, \tilde{X}_0(H_0))$, which is bounded around any point of $Z \setminus \partial\tilde{X}_0(H_0)$. Let f_Z be a continuous subanalytic function on $(Z \setminus Z_1, \tilde{X}_0(H_0))$, which is bounded around any point of $Z_1 \setminus \tilde{X}_0(H_0)$.

Proposition 8.6 *There exists $m_1 \geq m_0$ such that the following holds for any $m \geq m_1$:*

- *Suppose that $\mathcal{Z}_m \neq \emptyset$. Then, the restriction of $\tilde{\psi}_m^*(f)$ to a connected component of $\mathcal{V}_m \cap \tilde{\psi}_m^{-1}(\tilde{X}_0(H_0) \setminus Z)$ is described as a sum of a bounded function and a ramified analytic function of the form $\sum_{\eta < 0} f_{m,\eta}(\theta_m) \rho_m^\eta$. Moreover, the function $\tilde{\psi}_m^*(f_Z)$ on \mathcal{Z}_m is described as a sum of a bounded function and a ramified analytic function of the form $\sum_{\eta < 0} f_{Z,m,\eta}(\theta_m) \rho_m^\eta$ on $\mathcal{V}_{m,1}$ or $\mathcal{V}_{m,2}$ under the isomorphism (34) or (35).*
- *Suppose that $\mathcal{Z}_m = \emptyset$. Then, we have a (possibly empty) closed subanalytic subset $\mathcal{A}_m \subset \mathcal{V}_m$ such that the following holds:*
 - *Set $\mathcal{A}_m^\circ := \mathcal{A}_m \setminus \varpi_m^{-1}(H_m)$. If $\mathcal{A}_m \neq \emptyset$, we have one of the isomorphisms $\mathcal{A}_m \simeq \mathcal{V}_{m,1}$ or $\mathcal{A}_m \simeq \mathcal{V}_{m,2}$ induced by $u_m \mapsto a_m$ or $u_m \mapsto b_m$.*
 - *The restriction of $\tilde{\psi}_m^*(f)$ to a connected component of $\mathcal{V}_m \setminus (\mathcal{A}_m \cup \varpi_m^{-1}(H_m))$ is described as a sum of a bounded function and a ramified analytic function of the form $\sum_{\eta < 0} f_{m,\eta}(\theta_m) \rho_m^\eta$. Moreover, the restriction of $\tilde{\psi}_m^*(f)$ to \mathcal{A}_m° is described as a sum of a bounded function and a ramified analytic function of the form $\sum_{\eta < 0} f_{Z,m,\eta}(\theta_m) \rho_m^\eta$ on $\mathcal{V}_{m,1}$ or $\mathcal{V}_{m,2}$ under the isomorphism (34) or (35).*

8.2 The case $\hat{\kappa}(\mathbf{Y}) < \infty$

8.2.1 Proof of Proposition 8.1

In the following, we shall make \mathcal{I}_m and r_m smaller. Let \mathcal{V}_m denote a small neighbourhood of $\mathcal{I}_m \times [0, r_m[$ in $\tilde{X}_m(H_m)$ of the form $\mathcal{U}_m \times \{\theta_1 < \theta < \theta_2\} \times [0, r_m[$. We denote m_i by $m(i)$.

Because of the existence of $\tilde{\psi}_{m(i),0}$, we have $\tilde{\psi}_{m(i)}(\mathcal{I}_{m(i)} \times [0, r_{m(i)}]) \subset U^{(0)}$ if $r_{m(i)}$ are sufficiently small. Hence, we may assume that $\tilde{\psi}_{m(i)}(\mathcal{V}_{m(i)}) \subset U^{(0)}$. Recall that $C^{(0)} \subset U^{(0)}$ is given as $\{g_1^{(0)} = \dots = g_{r(0)}^{(0)} = 0\}$. We use the change of parametrization in §5.3.2, and we apply Proposition 7.20. Then, by shrinking $\mathcal{I}_{m(i)}$, $r_{m(i)}$ and $\mathcal{V}_{m(i)}$, we have $m^{(0)}$ such that for any $m(i) \geq m^{(0)}$ the following holds: (i) $\tilde{\psi}_{m(i)}^*(g_j^{(0)})$ are not constantly 0, (ii) the order $\text{ord}_{\rho_{m(i)}} \tilde{\psi}_{m(i)}^*(g_j^{(0)})(u_{m(i)}, \theta_{m(i)}, \rho_{m(i)})$ are constant with respect to $(u_{m(i)}, \theta_{m(i)}) \in \varpi_{m(i)}^{-1}(H_{m(i)}) \cap \mathcal{V}_{m(i)}$. Then, we have a unique morphism $\tilde{\psi}_{m(i)}^{(1)} : \mathcal{V}_{m(i)} \rightarrow W^{(1)}$ such that $\phi^{(1)} \circ \tilde{\psi}_{m(i)}^{(1)}$ is equal to $\tilde{\psi}_{m(i)}$ for any $m(i) \geq m^{(0)}$. By taking a subsequence, we may assume to have such morphisms $\tilde{\psi}_{m(i)}^{(1)}$ for any $m(i)$.

Suppose that for $\ell \geq 1$ we have already constructed real analytic morphisms $\tilde{\psi}_{m(i)}^{(\ell)} : \mathcal{V}_{m(i)} \rightarrow W^{(\ell)}$ such that $\xi^{(\ell)} \circ \tilde{\psi}_{m(i)}^{(\ell)} = \tilde{\psi}_{m(i)}$. Because of the existence of $\tilde{\psi}_{m(i),0}$, we may assume that $\text{Im } \tilde{\psi}_{m(i)}^{(\ell)}$ is contained in $U^{(\ell)}$, after making $\mathcal{V}_{m(i)}$ smaller. We obtain real analytic functions $(\tilde{\psi}_{m(i)}^{(\ell)})^*(g_j^{(\ell)})$ on $\mathcal{V}_{m(i)}$, and the real analytic functions $(\tilde{\psi}_{m(i)}^{(\ell)})^*(g_j^{(\ell)})$ ($i \geq 1$) are obtained as the pull back of $(\tilde{\psi}_{m(1)}^{(\ell)})^*(g_j^{(\ell)})$ by $\tilde{\psi}_{m(1),m(i)}$. Hence, by using the change of parametrization as in §5.3.2 and Proposition 7.20, and by shrinking $\mathcal{I}_{m(i)}$, $r_{m(i)}$ and $\mathcal{V}_{m(i)}$, we can take $m^{(\ell)}$ such that for any $m(i) \geq m^{(\ell)}$ the following holds: (i) $\tilde{\psi}_{m(i)}^*(g_j^{(\ell)})$ are not constantly 0, (ii) the orders $\text{ord}_{\rho_{m(i)}} \tilde{\psi}_{m(i)}^*(g_j^{(\ell)})(u_{m(i)}, \theta_{m(i)}, \rho_{m(i)})$ are constant with respect to $(u_{m(i)}, \theta_{m(i)}) \in \varpi_{m(i)}^{-1}(H_{m(i)}) \cap \mathcal{V}_{m(i)}$. Hence, we have a unique morphism $\tilde{\psi}_{m(i)}^{(\ell+1)} : \mathcal{V}_{m(i)} \rightarrow W^{(\ell+1)}$ such that $\phi^{(\ell+1)} \circ \tilde{\psi}_{m(i)}^{(\ell+1)} = \tilde{\psi}_{m(i)}^{(\ell)}$. In this way, the inductive construction can proceed, and we obtain Proposition 8.1 in the case $\hat{\kappa}(\mathbf{Y}) < \infty$. \blacksquare

8.2.2 Proof of Proposition 8.4

We take a rectilinearization $(W_\lambda, \phi_\lambda)$ ($\lambda \in \Lambda$) for f . By Proposition 8.1 and Proposition 8.2, we have $\lambda_0 \in \Lambda$, a positive number $m(1)$, an open subset $\mathcal{V}_{m(1)}$ in $\tilde{X}_{m(1)}(H_{m(1)})$ such that $\mathcal{V}_{m(1)} \cap \varpi_{m(1)}^{-1}(P_{m(1)}) \neq \emptyset$, and a real analytic morphism $\hat{\psi}_{m(1)} : \mathcal{V}_{m(1)} \rightarrow W_{\lambda_0}$ such that $\phi_{\lambda_0} \circ \hat{\psi}_{m(1)} = \tilde{\psi}_{m(1)|\mathcal{V}_{m(1)}}$. We take a sequence of open subsets $\mathcal{V}_m \subset \tilde{X}_m(H_m)$ ($m \geq m(1)$) such that (i) $\mathcal{V}_m \cap \varpi_m^{-1}(P_m) \neq \emptyset$, (i) $\tilde{\psi}_{m',m}(\mathcal{V}_m) \subset \mathcal{V}_{m'}$ for any $m \geq m' \geq m(1)$.

The set $\phi_{\lambda_0}^{-1}(Z)$ is expressed as the 0-set of a real analytic function h_{λ_0} on W_{λ_0} . The set $\tilde{\psi}_{m(1)}^{-1}(Z)$ is contained in the 0-set of $h_{m(1),\lambda_0} := \hat{\psi}_{m(1)}^*(h_{\lambda_0})$. If m is sufficiently larger than $m(1)$, after shrinking \mathcal{V}_m , we obtain that the 0-set of $\tilde{\psi}_{m(1),m}^*(h_{m(1),\lambda_0})$ is contained in $\mathcal{V}_m \cap \varpi_m^{-1}(H_m)$ by Corollary 7.21. It implies the first claim of Proposition 8.4 in the case $\hat{\kappa}(\mathbf{Y}) < \infty$.

Let (x_1, x_2, x_3, x_4) be the coordinate system on W_{λ_0} . We obtain the real analytic functions $\hat{\psi}_{m(1)}^*(x_i)$. After replacing $m(1)$ with a larger number, we may assume that $x_{i,m(1)} := \hat{\psi}_{m(1)}^*(x_i) > 0$ on $\mathcal{V}_{m(1)}$. If m is sufficiently larger than $m(1)$, after shrinking \mathcal{V}_m , we obtain that the order $\text{ord}_{\rho_m} \tilde{\psi}_{m(1),m}^*(x_{i,m(1)})(u_m, \theta_m, \rho_m)$ are constant with respect to (u_m, θ_m) . Then, $\tilde{\psi}_{m(1),m}^*(x_{i,m(1)})^{1/e}$ are ramified analytic functions on \mathcal{V}_m .

We have the description $\tilde{\psi}_m^*(f) = \tilde{\psi}_{m(1),m}^* \hat{\psi}_{m(1)}^*(\phi_{\lambda_0}^*(f))$, and the function $\phi_{\lambda_0}^*(f)$ is expressed as analytic functions of $(x_1^{1/e}, x_2^{1/e}, x_3^{1/e}, x_4^{1/e})$ for some $e \in \mathbb{Z}_{>0}$. Hence, we obtain that $\tilde{\psi}_m^*(f)$ is a ramified analytic function on \mathcal{V}_m if m is sufficiently larger than $m(1)$, i.e., we obtain the second claim of Proposition 8.4 in the case $\hat{\kappa}(\mathbf{Y}) < \infty$.

By replacing $m(1)$ with a larger number, we may assume that $\tilde{\psi}_m^*(f)$ is a ramified analytic function. Then, by Corollary 7.21 we obtain the third claim of Proposition 8.4 in the case $\hat{\kappa}(\mathbf{Y}) < \infty$. Thus, the proof of Proposition 8.4 is finished in the case $\hat{\kappa}(\mathbf{Y}) < \infty$. \blacksquare

8.3 The case $\hat{\kappa}(\mathbf{Y}) = \infty$

8.3.1 Change of parametrization

As in §5.3.2, we change the parametrization of curves. Let \mathcal{U}_m denote a small neighbourhood of 0 in \mathbb{C}_{u_m} . We have the holomorphic embedding $\Phi_m^{-1} : \mathcal{U}_m \times \Delta_{y^{1/\delta(m)}, \epsilon_m} \rightarrow U_m$, and the composition $F_m = \psi_m \circ \Phi_m^{-1}$ is described as

$$F_m(u_m, y^{1/\delta(m)}) = (g_{\mathbf{Y}_m}(u_m, y^{1/\delta(m)}), y).$$

We have the induced morphisms $\tilde{F}_m : \mathcal{U}_m \times \tilde{\Delta}_{y^{1/\delta(m)}, \epsilon_m}(0) \rightarrow \tilde{X}_0(H_0)$. It is enough to study \tilde{F}_m instead of $\tilde{\psi}_m$.

We use the polar coordinate $y = te^{\sqrt{-1}\phi}$, i.e., $y^{1/\delta(m)} = t^{1/\delta(m)} e^{\sqrt{-1}\phi/\delta(m)}$. We set $\mathfrak{J} := \{0 \leq \phi \leq 2\pi\}$. We have the map $\mathcal{U}_m \times \mathfrak{J} \times \{0 \leq t^{1/\delta(m)} \leq \epsilon\} \rightarrow \tilde{X}_0(H_0)$ given by $(u_m, \phi, t^{1/\delta(m)}) \mapsto \tilde{F}_m(u_m, e^{\sqrt{-1}\phi/\delta(m)} t^{1/\delta(m)})$. We may regard $\mathcal{I}_m \subset \mathfrak{J}$. The induced $\mathcal{N}_{\mathcal{U}_m \times \mathfrak{J}}$ -paths in $\tilde{X}_0(H_0)$ are also denoted by \tilde{F}_m . (See §7.3 for \mathcal{N} -path.)

8.3.2 Limit

We have the expansions $g_{\mathbf{Y}_m}(u_m, t^{1/\delta(m)} e^{\sqrt{-1}\phi/\delta(m)}) = \sum_{\eta} g_{\mathbf{Y}_m, \eta}(u_m) e^{\sqrt{-1}\phi\eta} t^{\eta}$. Recall that for $\eta < \kappa(\mathbf{Y}_m)$, the coefficients $g_{\mathbf{Y}_m, \eta}(u_m)$ are independent of $m' > m$, and constant with respect to u_m . We define $g_{\mathbf{Y}, \eta}$ as $g_{\mathbf{Y}_m, \eta} \in \mathbb{C}$ ($\eta < \kappa(\mathbf{Y}_m)$). We set $g_{\mathbf{Y}} := \sum g_{\mathbf{Y}, \eta} e^{\sqrt{-1}\phi\eta} t^{\eta}$ as a section of $\mathcal{N}_{\mathfrak{J}}$. (See §7.3 for the sheaf $\mathcal{N}_{\mathfrak{J}}$ on \mathfrak{J} .) We obtain the non-convergent $\mathcal{N}_{\mathfrak{J}}$ -path \tilde{F}_{∞} in $\tilde{X}_0(H_0)$ given by $\tilde{F}_{\infty}(\phi, t) = (g_{\mathbf{Y}}, \phi, t)$. We can naturally regard \tilde{F}_{∞} as a $\mathcal{N}_{\mathcal{U}_m \times \mathfrak{J}}$ -path in $\tilde{\mathbb{C}}^2(H)$ for each m . By the construction, we have $\text{ord}_t(\tilde{F}_{\infty}, \tilde{F}_m) \geq \kappa(\mathbf{Y}_m)$.

8.3.3 Proof of Proposition 8.1

Recall that for any \mathcal{N}_Y -path φ in a space B we denote the underlying map $Y \rightarrow B$ by φ_0 .

We set $\mathfrak{J}^{(0)'} := ((\tilde{F}_{\infty})_0)^{-1}(U^{(0)})$. We have the induced $\mathcal{N}_{\mathfrak{J}^{(0)'}}$ -path $\tilde{F}_{\infty| \mathfrak{J}^{(0)'}}$ in $U^{(0)}$. Note that $\tilde{F}_{\infty| \mathfrak{J}^{(0)'}}$ does not factor through $C^{(0)}$ by Lemma 7.25. We have the 0-dimensional closed subset $Z^{(0)} \subset \mathfrak{J}^{(0)'}$ such that for any $P \in \mathfrak{J}^{(0)'} \setminus Z^{(0)}$ we have

$$\text{ord}_{P,t}(\tilde{F}_{\infty| \mathfrak{J}^{(0)'}}(C^{(0)})) = \text{ord}_{P,t}(\tilde{F}_{\infty| P}(C^{(0)})) =: \mu^{(0)}.$$

We set $\mathfrak{J}^{(1)} := \mathfrak{J}^{(0)'} \setminus Z^{(0)}$. As explained in §7.3.3, we have the $\mathcal{N}_{\mathfrak{J}^{(1)}}$ -path $\tilde{F}_{\infty, \mathfrak{J}^{(1)}}^{(1)}$ in $W^{(1)}$ which is the lift of $\tilde{F}_{\infty| \mathfrak{J}^{(1)}}$, i.e., $\phi^{(1)} \circ \tilde{F}_{\infty, \mathfrak{J}^{(1)}}^{(1)} = \tilde{F}_{\infty| \mathfrak{J}^{(1)}}$.

We continue such a process inductively, as possible. Suppose that we have already constructed an $\mathcal{N}_{\mathfrak{J}^{(\ell)}}$ -path $\tilde{F}_{\infty, \mathfrak{J}^{(\ell)}}^{(\ell)}$ in $W^{(\ell)}$ for $\ell \geq 1$. We set $\mathfrak{J}^{(\ell)'} := \left((\tilde{F}_{\infty, \mathfrak{J}^{(\ell)}}^{(\ell)})_0 \right)^{-1} (U^{(\ell)})$. If $\mathfrak{J}^{(\ell)'} \neq \emptyset$, we have the induced $\mathcal{N}_{\mathfrak{J}^{(\ell)'}}$ -path $\tilde{F}_{\infty, \mathfrak{J}^{(\ell)'}}^{(\ell)}$ in $U^{(\ell)}$. Note that $\tilde{F}_{\infty, \mathfrak{J}^{(\ell)'}}^{(\ell)}$ does not factor through $C^{(\ell)}$. We have a 0-dimensional subset $Z^{(\ell)} \subset \mathfrak{J}^{(\ell)'}$ such that for any $P \in \mathfrak{J}^{(\ell)'} \setminus Z^{(\ell)}$ we have

$$\text{ord}_{P,t} \left(\tilde{F}_{\infty, \mathfrak{J}^{(\ell)'}}^{(\ell)}, C^{(\ell)} \right) = \text{ord}_{P,t} \left((\tilde{F}_{\infty, \mathfrak{J}^{(\ell)'}}^{(\ell)})|_P, C^{(\ell)} \right) =: \mu^{(\ell)}.$$

We set $\mathfrak{J}^{(\ell+1)} := \mathfrak{J}^{(\ell)'} \setminus Z^{(\ell)}$. We have the $\mathcal{N}_{\mathfrak{J}^{(\ell+1)}}$ -path $\tilde{F}_{\infty, \mathfrak{J}^{(\ell+1)}}^{(\ell)}$ in $W^{(\ell+1)}$ which is the lift of $\tilde{F}_{\infty, \mathfrak{J}^{(\ell+1)}}^{(\ell)}$.

We continue the process until either $\ell = k$ or $\mathfrak{J}^{(\ell)'} = \emptyset$ holds.

We set $N_0 := \sum_{\ell} \mu^{(\ell)} < \infty$. We can easily obtain the claim of Proposition 8.1 from the following lemma, in the case $\hat{\kappa}(\mathbf{Y}) = \infty$ for non-convergent \mathbf{Y} .

Lemma 8.7

- We have an open subset $\mathfrak{J}_0 \subset \mathfrak{J}$ with $\mathfrak{J}_0 \cap \mathcal{I}_{m_i} \neq \emptyset$ ($i \geq 1$) and an $\mathcal{N}_{\mathfrak{J}_0}$ -path $\hat{F}_{\infty, \mathfrak{J}_0}$ in W which is a lift of $\tilde{F}_{\infty, \mathfrak{J}_0}$ with respect to ϕ .
- Take any $N_1 > 0$. Let A be any complex manifold. Let ν be any $\mathcal{N}_{\mathfrak{J}_0 \times A}$ -path in $\tilde{X}_0(H_0)$ such that

$$\text{ord}_{t,P}(\nu|_{\mathfrak{J}_0 \times \{a\}}, \tilde{F}_{\infty, \mathfrak{J}_0}) > N_0 + N_1$$

for any $P \in \mathfrak{J}_0$ and for any $a \in A$. Then, we have the $\mathcal{N}_{\mathfrak{J}_0 \times A}$ -path $\tilde{\nu}$ in W such that (i) it is the lift of ν , (ii) $\text{ord}_{t,P}(\tilde{\nu}|_{\mathfrak{J}_0 \times \{a\}}, \hat{F}_{\infty, \mathfrak{J}_0}) > N_1$ for any $P \in \mathfrak{J}_0$ and $a \in A$.

- Take any $N_1 > 0$. If m_i is sufficiently large, we have the $\mathcal{N}_{\mathcal{U}_m \times (\mathfrak{J}_0 \cap \mathcal{I}_m)}$ -path \hat{F}_{m_i} in W such that (i) \hat{F}_{m_i} is the lift of $\tilde{F}_{m_i|_{\mathcal{U}_m \times (\mathfrak{J}_0 \cap \mathcal{I}_m)}}$ with respect to ϕ , (ii) we have

$$\text{ord}_{t,P}(\hat{F}_{\infty, \mathfrak{J}_0}, \hat{F}_{m_i|_{\{R\} \times (\mathfrak{J}_0 \cap \mathcal{I}_m)}}) > N_1$$

at any $P \in \mathfrak{J}_0 \cap \mathcal{I}_m$ and any $R \in \mathcal{U}_{m_i}$.

Proof If m_i is sufficiently large, we have $\text{ord}(\tilde{F}_{m_i, \mathfrak{J}}, \tilde{F}_{\infty, \mathfrak{J}}) > N_0$. Then, by an inductive argument on ℓ , we can show that (i) $\mathfrak{J}^{(\ell)} \neq \emptyset$, (ii) $\mathfrak{J}_{m_i}^{(\ell)} := \mathcal{I}_{m_i} \cap \mathfrak{J}^{(\ell)} \neq \emptyset$, (iii) we have

$$\text{ord}_{t,P} \left(\tilde{F}_{m_i, \mathfrak{J}_{m_i}^{(\ell)}}^{(\ell)}, \tilde{F}_{\infty, \mathfrak{J}_{m_i}^{(\ell)}}^{(\ell)} \right) > N_0 - \sum_{p \leq \ell} \mu^{(p)}.$$

Hence, we obtain the first claim. We can also prove the second claim similarly by an easy induction. The third claim follows from the second. Thus, we obtain Lemma 8.7 and Proposition 8.1. \blacksquare

8.3.4 Pull back of analytic functions

We make a preliminary for the proof of Propositions 8.5 and 8.6. Recall that (x, y) is the coordinate of $X_0 = \mathbb{C}^2$ such that $H_0 = \{y = 0\}$. Let $y = re^{\sqrt{-1}\theta}$ be the polar decomposition. Take an interval $\mathfrak{J} := \{\theta_1 < \theta < \theta_2\} \subset \mathfrak{J}$. We consider an open subset $\mathcal{B} := \{(x, \theta, r) \mid |x| < \epsilon_1, \theta_1 < \theta < \theta_2, 0 \leq r < \epsilon_2\}$ in $\tilde{X}_0(H_0)$. Suppose that we are given an analytic function h on \mathcal{B}_m which is not constant. We assume the following.

- The exterior derivative of h is not 0 at every point of $h^{-1}(0) \cap \{r > 0\}$.
- $\tilde{F}_{\infty, \mathfrak{J}}^*(h) = 0$ as a section of $\mathcal{N}_{\mathfrak{J}}$.

Let $x = a + \sqrt{-1}b$ be the real coordinate.

Lemma 8.8 *One of $\tilde{F}_{\infty, \mathfrak{J}}^*(\partial_a h)$ or $\tilde{F}_{\infty, \mathfrak{J}}^*(\partial_b h)$ is not 0.*

Proof Because $\tilde{F}_{\infty, \mathfrak{J}}^*(h) = 0$, we have $\tilde{F}_{\infty, \mathfrak{J}}^*(dh) = 0$, and hence the following equalities:

$$\tilde{F}_{\infty, \mathfrak{J}}^*(\partial_a h) \partial_t \tilde{F}_{\infty, \mathfrak{J}}^*(a) + \tilde{F}_{\infty, \mathfrak{J}}^*(\partial_b h) \partial_t \tilde{F}_{\infty, \mathfrak{J}}^*(b) + \tilde{F}_{\infty, \mathfrak{J}}^*(\partial_r h) = 0 \quad (36)$$

$$\tilde{F}_{\infty, \mathfrak{J}}^*(\partial_a h) \partial_\phi \tilde{F}_{\infty, \mathfrak{J}}^*(a) + \tilde{F}_{\infty, \mathfrak{J}}^*(\partial_b h) \partial_\phi \tilde{F}_{\infty, \mathfrak{J}}^*(b) + \tilde{F}_{\infty, \mathfrak{J}}^*(\partial_\theta h) = 0 \quad (37)$$

By the assumption, at least one of $\tilde{F}_{\infty, \mathfrak{J}}^*(\partial_\kappa h)$ ($\kappa = a, b, \theta, r$) is not 0. If we have $\tilde{F}_{\infty, \mathfrak{J}}^*(\partial_a h) = 0$ and $\tilde{F}_{\infty, \mathfrak{J}}^*(\partial_b h) = 0$, then we obtain $\tilde{F}_{\infty, \mathfrak{J}}^*(\partial_\kappa h)$ ($\kappa = \theta, r$) by the above equalities (36) and (37). Hence, we obtain the claim of the lemma. \blacksquare

Set \mathfrak{k} denote the minimum of $\text{ord}_t \tilde{F}_{\infty, \mathfrak{J}}^*(\partial_a h)$ and $\text{ord}_t \tilde{F}_{\infty, \mathfrak{J}}^*(\partial_b h)$. Note that we have the expansion $g_{\mathbf{Y}_m} = \sum_\eta g_{\mathbf{Y}_m, \eta}(u_m) e^{\sqrt{-1}\phi\eta} t^\eta$, and that the coefficients $g_{\mathbf{Y}_m, \eta}$ ($\eta < \kappa(\mathbf{Y}_m)$) are constants and independent of m .

Lemma 8.9 *Take any m such that $\kappa(\mathbf{Y}_m) > \mathfrak{k}$. Then, we have a non-empty interval $\mathfrak{J}_m \subset \mathfrak{J}$, a small neighbourhood \mathcal{U}_m and $\epsilon_m > 0$ such that either one of $\tilde{F}_{m, \mathcal{U}_m \times \mathfrak{J}_m}^*(\partial_a h)$ or $\tilde{F}_{m, \mathcal{U}_m \times \mathfrak{J}_m}^*(\partial_b h)$ is nowhere vanishing on $\mathcal{U}_m \times \mathfrak{J}_m \times \{0 < t < \epsilon_m\}$.*

Proof We have $\tilde{F}_{m, \mathcal{U}_m \times \mathfrak{J}_m}^*(\partial_\kappa h) \equiv \tilde{F}_{\infty, \mathfrak{J}_m}^*(\partial_\kappa h)$ ($\kappa = a, b$) modulo $t^{\kappa(\mathbf{Y}_m)}$. Hence, we have either one of $\text{ord}_t \tilde{F}_{m, \mathcal{U}_m \times \mathfrak{J}_m}^*(\partial_a h) = \text{ord}_t \tilde{F}_{\infty, \mathfrak{J}_m}^*(\partial_a h) = \mathfrak{k}$ or $\text{ord}_t \tilde{F}_{m, \mathcal{U}_m \times \mathfrak{J}_m}^*(\partial_b h) = \text{ord}_t \tilde{F}_{\infty, \mathfrak{J}_m}^*(\partial_b h) = \mathfrak{k}$. Then, we obtain the claim of the lemma. \blacksquare

Note that $g_{\mathbf{Y}_m, \kappa(\mathbf{Y}_m)}(u_m) - g_{\mathbf{Y}_m, \kappa(\mathbf{Y}_m)}(0)$ are \mathbb{C} -affine functions of u_m . Let (a_m, b_m) be as in §8.1.5. We obtain the following.

Lemma 8.10 *Take a large m such that $\kappa(\mathbf{Y}_m) > \mathfrak{k}$. Then, we have a non-empty interval $\mathfrak{J}_m \subset \mathfrak{J}$, a small neighbourhood \mathcal{U}_m and $\epsilon_m > 0$ such that either $\partial_{a_m} \tilde{F}_{m, \mathcal{U}_m \times \mathfrak{J}_m}^*(h)$ or $\partial_{b_m} \tilde{F}_{m, \mathcal{U}_m \times \mathfrak{J}_m}^*(h)$ is nowhere vanishing on $\mathcal{U}_m \times \mathfrak{J}_m \times \{0 < t < \epsilon_m\}$. As a result, either one of the following holds.*

- We have $\tilde{F}_{m, \mathcal{U}_m \times \mathfrak{J}_m}^*(h)^{-1}(0) \cap \{t > 0\} \simeq \{|a_m| < \delta_{1,m}\} \times \mathfrak{J}_m \times \{0 < t < \epsilon_m\}$ induced by $u_m \mapsto a_m$.
- We have $\tilde{F}_{m, \mathcal{U}_m \times \mathfrak{J}_m}^*(h)^{-1}(0) \cap \{t > 0\} \simeq \{|b_m| < \delta_{2,m}\} \times \mathfrak{J}_m \times \{0 < t < \epsilon_m\}$ induced by $u_m \mapsto b_m$. \blacksquare

8.3.5 Proof of Propositions 8.5 and 8.6

We take a rectilinearization $\phi_\lambda : W_\lambda \rightarrow \mathcal{Y}$ ($\lambda \in \Lambda$) of f and f_Z . (See §8.1.2 for \mathcal{Y} .) The set $\phi_\lambda^{-1}(Z)$ are the union of some tuples of quadrants $Q \subset W_\lambda$ with $\dim Q \leq 3$. Let $Q_{\lambda, q} \subset W_\lambda$ be a 4-dimensional quadrant such that $Q_{\lambda, q} \subset \phi_\lambda^{-1}(\tilde{X}_0(H_0) \setminus Z)$. We have

$$\phi_\lambda^*(f)|_{Q_{\lambda, q}} = a(\lambda, q) \cdot \prod_{p=1}^4 (\pm x_p)^{\ell_p(\lambda, q)/\rho(\lambda, q)},$$

where $a(\lambda, q)$ is a nowhere vanishing ramified analytic function, and $\rho(\lambda, q) \in \mathbb{Z}_{>0}$, and the signature of $\pm x_p$ are chosen so that $\pm x_p > 0$ on $Q_{\lambda, q}$. We have either (i) $\ell_p(\lambda, q) \geq 0$ for any p , or (ii) $\ell_p(\lambda, q) \leq 0$ for any p . If $\partial Q_{\lambda, q} \cap \phi_\lambda^{-1}(X_0 \setminus H_0) \neq \emptyset$, then $\phi_\lambda^*(f)|_{Q_{\lambda, q}}$ is bounded around any point of $\partial Q_{\lambda, q} \cap \phi_\lambda^{-1}(X_0 \setminus H_0)$. Hence, we have the restriction of $\phi_\lambda^*(f)|_{Q_{\lambda, q}}$ to any quadrant contained in $\partial Q_{\lambda, q} \cap \phi_\lambda^{-1}(X_0 \setminus H_0)$.

By Proposition 8.1, we have λ_0 , an interval $\mathfrak{J}_1 \subset \mathfrak{J}$, and $\mathcal{N}_{\mathcal{U}_m \times \mathfrak{J}_1}$ -paths \hat{F}_m in W_{λ_0} for any large m , which are lifts of the restriction of \tilde{F}_m . We also have the $\mathcal{N}_{\mathfrak{J}_1}$ -path $\hat{F}_{\infty, \mathfrak{J}_1}$ in W_{λ_0} which is the lift of the restriction of \tilde{F}_∞ . For any $N_1 > 0$, we have k_0 such that the following holds for any $k \geq k_0$ and for any $(R_m, P) \in \mathcal{U}_m \times \mathfrak{J}_1$:

$$\text{ord}_{P, t}(\hat{F}_{\infty, \mathfrak{J}_1|P}, \hat{F}_m|_{(R_m, P)}) > N_1. \quad (38)$$

We set $I_{t^{1/\delta(m)}} = \{0 \leq t^{1/\delta(m)} \leq \epsilon_m\}$. We regard the $\mathcal{N}_{\mathcal{U}_m \times \mathfrak{J}_1}$ -path $\tilde{F}_m|_{\mathcal{U}_m \times \mathfrak{J}_1}$ as an analytic map

$$\tilde{F}_{m, \mathcal{U}_m \times \mathfrak{J}_1} : \mathcal{U}_m \times \mathfrak{J}_1 \times I_{t^{1/\delta(m)}} \longrightarrow \tilde{X}_0(H_0).$$

Let $Z^{(m)} \subset \mathcal{U}_m \times \mathfrak{J}_1 \times I_{t^{1/\delta(m)}}$ be the pull back of Z by $\tilde{F}_{m, \mathcal{U}_m \times \mathfrak{J}_1}$. We have $\dim_{\mathbb{R}} Z^{(m)} \leq 3$.

Let $W_{\lambda_0}^{(\ell)}$ denote the union of the quadrants Q of W_{λ_0} with $\dim Q \leq \ell$. Let us consider the following three cases.

- (a1) $\hat{F}_{\infty, \mathfrak{J}_1}$ factors through $W_{\lambda_0}^{(2)}$.
- (a2) $\hat{F}_{\infty, \mathfrak{J}_1}$ factors through $W_{\lambda_0}^{(3)}$, but does not factor through $W_{\lambda_0}^{(2)}$.
- (a3) $\hat{F}_{\infty, \mathfrak{J}_1}$ does not factor through $W_{\lambda_0}^{(3)}$.

Let us observe that the case (a1) does not occur. Because the sheaf of rings $\mathcal{N}_{\mathfrak{J}_1}$ is integral, we obtain that $\hat{F}_{\infty, \mathfrak{J}_1}$ factors through a linear subspace $H \subset W_{\lambda_0}$ with $\dim H = 2$. Then $\hat{F}_{\infty, \mathfrak{J}_1}$ is convergent by Lemma 7.25, which contradicts with our assumption. Hence, $\hat{F}_{\infty, \mathfrak{J}_1}$ does not factor through $W_{\lambda_0}^{(2)}$.

Let us consider the case (a3). We have $\hat{F}_{\infty, \mathfrak{J}_1}^*(x_p) \neq 0$ for $p = 1, 2, 3, 4$. By shrinking \mathfrak{J}_1 , we may assume that $\text{ord}_t \hat{F}_{\infty, \mathfrak{J}_1|P}^*(x_p)$ are independent of $P \in \mathfrak{J}_1$. Then, if m is sufficiently large, we have $\text{ord}_t \hat{F}_{m|(u_m, P)}^*(x_p) = \text{ord}_t \hat{F}_{\infty, \mathfrak{J}_1|P}^*(x_p)$ for any $(u_m, P) \in \mathcal{U}_m \times \mathfrak{J}_1$. We obtain that $\hat{F}_m^{-1}(W_{\lambda_0}^{(3)}) \subset \mathcal{U}_m \times \mathfrak{J}_1 \times \{0\}$ for any sufficiently large m . It implies that $Z^{(m)} \setminus (\mathcal{U}_m \times \mathfrak{J}_1 \times \{0\}) = \emptyset$. Hence, the claims of Proposition 8.5 holds in the case (a3). Let us study the claim of Proposition 8.6 in the case (a3). We may assume that $(\hat{F}_m)^*(x_p) > 0$ ($p = 1, 2, 3, 4$) on $\mathcal{U}_m \times \mathfrak{J}_1 \times \{t^{1/\delta(m)} > 0\}$. Because of (38), the orders $\text{ord}_t (\hat{F}_m)^*(x_i)(u_m, \phi, t)$ are constant with respect to (u_m, ϕ) . Hence, $(\hat{F}_m)^*(x_p^{1/e})$ are ramified analytic on $\mathcal{U}_m \times \mathfrak{J}_1 \times I_{t^{1/\delta(m)}}$. We have the expansion

$$(\hat{F}_m)^*(f) = \sum_{\eta \geq -M} f_{\eta}^{(m)}(u_m, \phi) t^{\eta}.$$

By Lemma 8.11 below, if m is sufficiently large, $f_{\eta}^{(m)}(u_m, \phi)$ ($\eta < 0$) are independent of u_m . Hence, the claim of Proposition 8.6 holds with $\mathcal{Z}_m = \emptyset$ and $\mathcal{A}_m = \emptyset$.

Let us study the case (a2). We may assume that $\hat{F}_{\infty, \mathfrak{J}_1}$ factors through $\{x_1 = 0\}$. We have $\hat{F}_{\infty, \mathfrak{J}_1}^*(x_p) \neq 0$ ($p = 2, 3, 4$). By shrinking \mathfrak{J}_1 , we may assume that $\eta(p) := \text{ord}_t \hat{F}_{\infty, \mathfrak{J}_1|P}^*(x_p)$ ($p = 2, 3, 4$) are independent of P . We may assume that $\hat{F}_{\infty, \mathfrak{J}_1}^*(x_p)_{\eta(p)} > 0$ ($p = 2, 3, 4$) for the expansion $\hat{F}_{\infty, \mathfrak{J}_1}^*(x_p) = \sum \hat{F}_{\infty, \mathfrak{J}_1}^*(x_p)_{\eta} t^{\eta}$. We have m_{10} such that $\hat{F}_m^*(x_p) > 0$ ($p = 2, 3, 4$) for any $m \geq m_{10}$ on $\mathcal{U}_m \times \mathfrak{J}_1 \times (I_{t^{1/\delta(m)}} \setminus \{0\})$. Applying Lemma 8.10 to the pull back of $\hat{F}_{m_{10}}^*(x_1)$ via the induced morphisms $\mathcal{U}_m \times \mathfrak{J}_1 \times I_{t^{1/\delta(m)}} \longrightarrow \mathcal{U}_{m_{10}} \times \mathfrak{J}_1 \times I_{t^{1/\delta(m_{10})}}$ for any sufficiently large m , we obtain that $\hat{F}_{m_{10}}^*(x_1)^{-1}(0) \setminus (\mathcal{U}_m \times \mathfrak{J}_1 \times \{0\}) \simeq \mathcal{V}_{m,1}$ or $\hat{F}_{m_{10}}^*(x_1)^{-1}(0) \setminus (\mathcal{U}_m \times \mathfrak{J}_1 \times \{0\}) \simeq \mathcal{V}_{m,2}$ by the map $u_m \mapsto a_m$ or $u_m \mapsto b_m$. If $Z^{(m)} \setminus (\mathcal{U}_m \times \mathfrak{J}_1 \times \{0\}) \neq \emptyset$, it is equal to $\hat{F}_m^*(x_1)^{-1}(0) \setminus (\mathcal{U}_m \times \mathfrak{J}_1 \times \{0\})$. We also have $\hat{F}_m^{-1}(W_{\lambda_0}^{(2)}) \subset \mathcal{U}_m \times \mathfrak{J}_1 \times \{0\}$. Hence, we obtain that the claims of Proposition 8.5 hold in the case (a2).

Let us study the claims of Proposition 8.6 in the case (a2). We set $Q := \{x_1 = 0, x_2 > 0, x_3 > 0, x_4 > 0\}$, $Q_+ := \{x_1 > 0, x_2 > 0, x_3 > 0, x_4 > 0\}$ and $Q_- := \{x_1 < 0, x_2 > 0, x_3 > 0, x_4 > 0\}$. We set $f_{\pm} := \phi_{\lambda_0}^*(f)|_{Q_{\pm}}$. We may naturally regard f_{\pm} as the functions on $Q_{\pm} \cup Q$. In particular, we have the restrictions $f_{\pm|Q}$. Let $q_{\pm} : Q_{\pm} \longrightarrow Q$ be the projection forgetting x_1 . Then, we have the functions $q_{\pm}^*(f_{\pm|Q})$. We have $f_{\pm} - q_{\pm}^*(f_{\pm|Q}) = x_1^{\gamma_{\pm}} \cdot h_{\pm} \cdot \prod_{p=2}^4 x_p^{\ell_p/\rho}$ for some $\gamma_{\pm} > 0$, where h_{\pm} is a ramified analytic function on the closure of Q_{\pm} . We may assume that $\text{ord}_t \hat{F}_m^*(x_1)$ is sufficiently large. Hence, we obtain that $\hat{F}_m^*(f_{\pm}) - \hat{F}_m^*(q_{\pm}^* f_{\pm|Q})$ are bounded. By an argument as in the case (a3), we obtain that $\hat{F}_m^*(q_{\pm}^* f_{\pm|Q})$ is ramified analytic, and the coefficients of t^{η} ($\eta < 0$) depend only on θ_m .

Suppose $Z^{(m)} \setminus (\mathcal{U}_m \times \mathfrak{J}_1 \times \{0\}) \neq \emptyset$. We have $\phi_{\lambda_0}^*(f_Z) = a_Z \cdot \prod_{p=2}^4 x_p^{\ell_p(Z)/\rho(Z)}$, where a_Z is a ramified analytic function on \overline{Q} . By an argument as in the case (a3), we obtain that $\tilde{F}_m^*(f_Z)$ on $Z^{(m)} \setminus (\mathcal{U}_m \times \mathfrak{J}_1 \times \{0\})$ is ramified analytic as a function on $\mathcal{V}_{1,m}$ or $\mathcal{V}_{2,m}$, and the coefficients of t^{η} ($\eta < 0$) depend only on θ_m .

Suppose $Z^{(m)} \setminus (\mathcal{U}_m \times \mathfrak{J}_1 \times \{0\}) = \emptyset$. By an argument as in the case **(a3)**, we obtain that the restriction of $\widehat{F}_m^*(f|_Q)$ on $\widehat{F}_m^{-1}(Q) \setminus (\mathcal{U}_m \times \mathfrak{J}_1 \times \{0\})$ is ramified analytic as a function on $\mathcal{V}_{1,m}$ or $\mathcal{V}_{2,m}$, and the coefficients of t^η ($\eta < 0$) depend only on θ_m . Hence, the claims of Proposition 8.6 holds in the case **(a2)**. \blacksquare

8.3.6 Appendix

Let X be an open subset in \mathbb{R}^n . Let H be a hypersurface of X given by an analytic function $f = 0$. Let F be a subanalytic function on $(X \setminus H, \mathbb{R}^n)$. Let $\Phi : X' \rightarrow X$ be an analytic morphism such that $\Phi^*(f) = a \cdot \prod_{i=1}^n x_i^{m_i}$ and $\Phi^*(F) = b \cdot \prod_{i=1}^n x_i^{\ell_i}$ for a global coordinate system (x_1, \dots, x_n) of X' , where a is a nowhere vanishing analytic function, b is a nowhere vanishing ramified analytic function, and $(m_i) \in \mathbb{Z}_{\geq 0}^n$ and $(\ell_i) \in (\mathbb{Q}_{\geq 0})^n \cup (\mathbb{Q}_{\leq 0})^n$.

Let P_0 denote a one point set. Let γ' be an \mathcal{N}_{P_0} -path in X' such that $(\gamma')_0(P_0) = (0, \dots, 0)$. Let $\gamma = \Phi \circ \gamma'$ be the induced \mathcal{N}_{P_0} -path in X .

Lemma 8.11 *There exists an integer $N_0 > 0$ depending only on (ℓ_i) and $\text{ord } \gamma^*(f)$ such that the following holds.*

- Let $\sigma(t)$ be an \mathcal{N}_{P_0} -path in X' such that $\sigma_0(P_0) = (0, \dots, 0)$ and that $\text{ord}_t(\sigma, \gamma') = N_0$. Then, the polar parts of $(\gamma')^*F$ and σ^*F are the same.

Proof It is enough to consider the case $(\ell_i) \in \mathbb{Q}_{\leq 0}^n$. We have the description $\gamma'(t) = (\gamma'_1(t), \dots, \gamma'_n(t))$. Because $\gamma^*(f) = (\gamma')^*\Phi^*(f)$, we have $\text{ord}_t(\gamma'_i) \leq \text{ord}_t(\gamma^*(f))$. We set $d_i := \text{ord}_t(\gamma'_i)$ and $d := \text{ord}_t(\gamma^*f)$. We take $N_0 > nd + \sum_i |\ell_i|d$. We have the description $\sigma(t) = (\sigma_1, \dots, \sigma_n(t))$. We have $\sigma_i(t) = \gamma'_i(t) + t^{N_0}b_i(t)$. The following holds:

$$\prod_i \left(\gamma'_i(t) + t^{N_0}b_i(t) \right)^{\ell_i} = \prod_i \gamma'_i(t)^{\ell_i} (1 + t^{N_0-d_i}c_i(t))^{\ell_i} = \prod_i \gamma'_i(t)^{\ell_i} (1 + t^{N_0-d_i}c'_i(t)).$$

Then, the claim of the lemma is clear. \blacksquare

9 Meromorphic flat connections and enhanced ind-sheaves

9.1 Surface case

Let X be a complex surface with a normal crossing complex hypersurface H . Let $K \in \mathbb{E}_{\odot}^b(IC_{X(H)})$.

Proposition 9.1 *We have $K \in \mathbb{E}_{\text{mero}}^b(IC_{X(H)})$.*

Proof It is enough to prove the following for any $P \in H$:

$\mathcal{C}(P)$: There exists a neighbourhood U of P in X such that $K|_U \in \mathbb{E}_{\text{mero}}^b(IC_{U(H \cap U)})$.

Hence, we may assume that X is a relatively compact subset in another complex surface X' , and K is the restriction of an object defined on X' . By Proposition 4.13, Proposition 6.15 and Corollary 6.16, we have a projective morphism $\varphi : X_1 \rightarrow X$ such that (i) $H_1 := \varphi^{-1}(H)$ is normal crossing, (ii) $X_1 \setminus H_1 \simeq X \setminus H$, (iii) at any cross point of $P_1 \in H_1$, the condition $\mathcal{C}(P_1)$ for $\mathbb{E}\varphi^{-1}(K)$ is satisfied. Once we know that $\mathbb{E}\varphi^{-1}(K) \in \mathbb{E}_{\text{mero}}^b(IC_{X_1(H_1)})$, then we obtain that $K \in \mathbb{E}_{\text{mero}}^b(IC_{X(H)})$. Hence, we may assume that $\mathcal{C}(P)$ holds for any cross point P of H from the beginning.

Let P be a smooth point of H . Suppose that $\mathcal{C}(P)$ does not hold for K . We set $X^{(0)} := X$ and $P^{(0)} := P$. Let $\varphi^{(1)} : X^{(1)} \rightarrow X$ be the blowing up at P . We take a point $P^{(1)} \in (\varphi^{(1)})^{-1}(P^{(0)})$ such that $\mathcal{C}(P^{(1)})$ does not hold for $K^{(1)} := \mathbb{E}(\varphi^{(1)})^{-1}(K)$ if such a point $P^{(1)}$ exists. Inductively, we construct a sequence of morphisms

$$X^{(\ell)} \xrightarrow{\varphi^{(\ell)}} X^{(\ell-1)} \xrightarrow{\varphi^{(\ell-1)}} \dots \xrightarrow{\varphi^{(2)}} X^{(1)} \xrightarrow{\varphi^{(1)}} X^{(0)}$$

with points $P^{(i)} \in X^{(i)}$, such that (i) $\varphi^{(i)}$ is the blowing up of $X^{(i-1)}$ at $P^{(i-1)}$, (ii) $\mathcal{C}(P^{(i)})$ does not hold for $\mathbb{E}(\psi^{(i)})^{-1}(K)$ at $P^{(i)}$, where $\psi^{(i)} : X^{(i)} \rightarrow X$ is the induced morphism. We stop the process if $\mathcal{C}(Q)$ holds for $\mathbb{E}(\psi^{(\ell+1)})^{-1}(K)$ at any $Q \in (\varphi^{(\ell+1)})^{-1}(P^{(\ell)})$.

Lemma 9.2 *Suppose that we have already known that any such sequence will stop after a finite steps. Then, we have a number N such that any such sequence will stop after at most N -steps.*

Proof Suppose that we have a sequence $\ell_j \rightarrow \infty$, and sequences of morphisms

$$X_j^{(\ell_j)} \xrightarrow{\varphi_j^{(\ell_j)}} X_j^{(\ell_j-1)} \xrightarrow{\varphi_j^{(\ell_j-1)}} \dots \xrightarrow{\varphi_j^{(2)}} X_j^{(1)} \xrightarrow{\varphi_j^{(1)}} X_j^{(0)} = X$$

with points $P_j^{(i)} \in X_j^{(i)}$ as above. We shall derive a contradiction.

By the construction, we have $X_j^{(0)} = X$ and $P_j^{(0)} = P$. Hence, $X_j^{(1)}$ and $(\varphi_j^{(1)})^{-1}(P)$ are independent of j . We have a finite subset $D^{(1)} \subset (\varphi_j^{(1)})^{-1}(P)$ such that $\mathcal{C}(Q)$ holds for any $Q \in (\varphi_j^{(1)})^{-1}(P) \setminus D^{(1)}$. Hence, by going to a sub-sequence of (ℓ_j) , we may assume that $P_j^{(1)}$ are independent of j . Similarly, for any k_0 , after going to sub-sequences, we may assume that $X_j^{(k)}$ and $P_j^{(k)}$ ($k \leq k_0$) are independent of j . In this way, we obtain an infinite sequence of complex blowings up

$$\dots \longrightarrow X^{(i)} \xrightarrow{\varphi^{(i)}} X^{(i-1)} \xrightarrow{\varphi^{(i-1)}} \dots \longrightarrow X^{(1)} \xrightarrow{\varphi^{(1)}} X^{(0)} = X$$

with points $P^{(i)} \in X^{(i)}$ as above. It contradicts with the assumption. Thus, we obtain the claim of the lemma. \blacksquare

Let us observe that any such sequence will stop after a finite steps. Suppose that we have an infinite sequence, and we shall deduce a contradiction. By Corollary 6.16, it is described as an infinite sequence of complex blowings up associated to $\mathbf{Y} = (\boldsymbol{\eta}_1, \omega_1, \boldsymbol{\eta}_2, \omega_2, \dots) \in \prod_{i=1}^{\infty} (\{\pm\}^{\ell(i)} \times \mathbb{C}^*)$. We shall use the notation in §8.1.1.

We have a stratification $\tilde{X}(H) = \tilde{X}(H)^{(0)} \supset \tilde{X}(H)^{(1)} \supset \dots$ for K . We have $\tilde{X}(H) \setminus \tilde{X}(H)^{(1)} = \coprod \mathcal{C}_j$. We have subanalytic functions $g_{j,k}$ on $(\mathcal{C}_j, \tilde{X}(H))$ which control the growth order of $\pi^{-1}(\mathcal{C}_{C_j}) \otimes K$.

Suppose $\hat{\kappa}(\mathbf{Y}) < \infty$. We use Proposition 8.4. If m is sufficiently large, we have an open subset \mathcal{U} in $\tilde{X}_m(H_m)$ such that (i) $\mathcal{U} \cap \varpi_m^{-1}(P_m) \neq \emptyset$, (ii) $\tilde{\psi}_m(\mathcal{U} \setminus \varpi_m^{-1}(H_m)) \subset \mathcal{C}_{j_0}$ for some j_0 . Moreover, the functions $\tilde{\psi}_m^*(g_{j_0,k})$ and $\tilde{\psi}_m^*(g_{j_0,k_1} - g_{j_0,k_2})$ are ramified analytic, and the 0-sets of $\tilde{\psi}_m^*(g_{j_0,k})$ and $\tilde{\psi}_m^*(g_{j_0,k_1} - g_{j_0,k_2})$ are contained in $\varpi_m^{-1}(H_m)$ if the functions are not constantly 0. Moreover, $\text{ord}_{\rho_m} \tilde{\psi}_m^*(g_{j_0,k})(u_m, \theta_m, \rho_m)$ and $\text{ord}_{\rho_m} \tilde{\psi}_m^*(g_{j_0,k_1} - g_{j_0,k_2})(u_m, \theta_m, \rho_m)$ are independent of (u_m, θ_m) . Then, we obtain that K satisfies the condition **(GA)** at P_m . By Proposition 3.32, we obtain that $\mathcal{C}(P_m)$ holds. But, it contradicts with the assumption on the infinite sequence.

Suppose that \mathbf{Y} is not convergent and that $\hat{\kappa}(\mathbf{Y}) = \infty$. Let m be sufficiently large. Let \mathcal{I}_m be an interval in $\varpi_m^{-1}(P_m)$, and let \mathcal{V}_m be a neighbourhood of \mathcal{I}_m in $\tilde{X}_m(H_m)$ as in Propositions 8.5 and 8.6. Let us consider the case $\mathcal{Z}_m := \tilde{\psi}_m^{-1}(\tilde{X}(H)^{(1)}) \cap (\mathcal{V}_m \setminus \varpi_m^{-1}(H_m)) \neq \emptyset$. We explain only the case $\tilde{\psi}_m^{-1}(\tilde{X}(H)^{(1)}) \cap (\mathcal{V}_m \setminus \varpi_m^{-1}(H_m)) \simeq \mathcal{V}_{1,m}$ because the other case can be argued similarly. We have $\mathcal{V}_m \setminus (\varpi_m^{-1}(H_m) \cup \mathcal{Z}_m) = \mathcal{V}_{m,+} \sqcup \mathcal{V}_{m,-}$. Let $v_m = \rho_m e^{\sqrt{-1}\theta_m}$ be the polar decomposition. By Propositions 8.5 and 8.6, we have ramified analytic functions $h_1^{\pm}(\theta_m, \rho_m), \dots, h_k^{\pm}(\theta_m, \rho_m)$ such that $\pi^{-1}(\mathbb{C}_{\mathcal{V}_{m,\pm}}) \otimes K = \bigoplus \mathbb{C}^E \otimes \mathbb{C}_{t \geq h_j^{\pm}}^+$. We also have ramified analytic functions $h_1^0(\theta_m, \rho_m), \dots, h_k^0(\theta_m, \rho_m)$ such that $\pi^{-1}(\mathbb{C}_{\mathcal{Z}_m}) \otimes K = \bigoplus \mathbb{C}^E \otimes \mathbb{C}_{t \geq h_j^0}^+$. We may assume that h_j^{\pm} and h_j^0 have only negative powers of ρ_m . Recall that for a small neighbourhood \mathcal{B}_m of P_m in X_m there exists a meromorphic flat bundle (V, ∇) on $(\mathcal{B}_m \setminus P_m, (\mathcal{B}_m \cap H_m) \setminus P_m)$. Then, we can easily deduce $\{h_j^+\} = \{h_j^-\} = \{h_j^0\}$ by comparing them on $\{0 < a_m\} \times \mathcal{I}_m \times \{0 < \rho_m < \epsilon_m\}$. Then, we obtain that K satisfies the condition **(GA)**. By Proposition 3.32, we have $\text{DR}^E(V, \nabla)[-2] \simeq E\psi_m^{-1}(K)|_B$, i.e., $\mathcal{C}(P_m)$ holds for $E\psi_m^{-1}(K)$. But, it contradicts with the construction of the infinite sequence. The case $\mathcal{Z}_m = \emptyset$ can be argued similarly.

Suppose that \mathbf{Y} is convergent. By Lemma 5.5, after a finite step, each blowing up is taken at a smooth point. We may assume that each blowing up is taken at a smooth point from the beginning. By the convergence, we have a complex curve C in X such that (i) C is transversal with H , (ii) each blowing up is taken at the intersection of the exceptional fiber and the strict transform of C . We may assume that $X = \Delta^2$, $H = \{y = 0\}$

and $C = \{x = 0\}$. Then, we obtain that the process will stop after a finite step by Corollary 6.16. It contradicts with the construction of the infinite sequence. Thus, we obtain Proposition 9.1. \blacksquare

9.2 General case

Let X be any n -dimensional complex manifold with a complex hypersurface H . Let us prove the following theorem.

Theorem 9.3 *If $K \in E_{\odot}^b(IC_{X(H)})$, then $K \in E_{\text{mero}}^b(IC_{X(H)})$.*

It is enough to consider the case where H is a normal crossing hypersurface. Let (V_0, ∇) be a flat bundle on $X \setminus H$ with an isomorphism $\text{DR}(V_0, \nabla) \simeq K|_{X \setminus H}$.

9.2.1 Smooth case

Let us consider the case where $X = \Delta^n$ and $H = \{z_1 = 0\}$. According to Proposition 4.13, we have a closed subanalytic subset $Z \subset H$ with $\dim_{\mathbb{R}} Z \leq 2n - 4$, a good meromorphic flat bundle (V_1, ∇) on $(X \setminus Z, H \setminus Z)$, and an isomorphism $\text{DR}^E(V_1, \nabla) = K|_{X \setminus Z}$.

Let us observe that (V_1, ∇) is extended to a meromorphic flat connection on (X, H) . Recall that we have the Deligne-Malgrange lattice $V_1^{DM} \subset V_1$ which is a locally free $\mathcal{O}_{X \setminus Z}$ -submodule of V_1 such that $V_1^{DM}(* (H \setminus Z)) = V_1$. (See [25]. See also [28].)

We have a decomposition $Z = Z' \cup Z_1$, where Z' is smooth with $\dim_{\mathbb{R}} Z' = 2n - 4$, and $\dim_{\mathbb{R}} Z_1 \leq 2n - 5$. Let P be any point of Z'_1 . We have a small holomorphic coordinate system (X_P, w_1, \dots, w_n) around P such that (i) $X_P \simeq \Delta^n$ by the coordinate, (ii) $H \cap X_P = \{w_1 = 0\}$, (iii) $\max_{Z' \cap X_P} \{|w_2|\} \leq \epsilon < 1$. For any $\mathbf{b} = (w_3^0, \dots, w_n^0)$, set $X_{P,\mathbf{b}} = \{(w_1, w_2, w_3^0, \dots, w_n^0)\}$ and $H_{P,\mathbf{b}} := X_{P,\mathbf{b}} \cap H$. The restriction of V_1^{DM} to $X_{P,\mathbf{b}} \setminus Z'$ is the Deligne-Malgrange lattice of $V|_{X_{P,\mathbf{b}} \setminus Z'}$. By the result in the surface case (Proposition 9.1), $V_1|_{X \setminus Z'}$ is extended to a meromorphic flat connection on $(X_{P,\mathbf{b}}, H_{P,\mathbf{b}})$. Hence, $V_1^{DM}|_{X_{P,\mathbf{b}} \setminus Z'}$ is extended to a coherent $\mathcal{O}_{X_{P,\mathbf{b}}}$ -module. According to [41, Theorem 7.4], $V_1^{DM}|_{X_P \setminus Z'}$ is uniquely extended to a coherent reflexive \mathcal{O}_{X_P} -module. Hence, V_1 is extended to a coherent reflexive $\mathcal{O}_{X \setminus Z_1}(* (H \setminus Z_1))$ -module V_2 , i.e., (V_1, ∇) is extended to a meromorphic flat connection on $(X \setminus Z_1, H \setminus Z_1)$. We have the Deligne-Malgrange lattice V_2^{DM} of (V_1, ∇) , which is the canonical reflexive $\mathcal{O}_{X \setminus Z_1}$ -submodule of V_2 such that $V_2^{DM}(* (H \setminus Z_1)) = V_2$.

We have the decomposition $Z_1 = Z'_1 \cup Z_2$, where Z'_1 is smooth, and $\dim_{\mathbb{R}} Z_2 \leq 2n - 6$. Let P be any point of Z'_1 . We have a small holomorphic coordinate system (X_P, w_1, \dots, w_n) around P such that (i) $X_P \simeq \Delta^n$ by the coordinate, (ii) $H \cap X_P = \{w_1 = 0\}$, (iii) $\max_{Z'_1 \cap X_P} \{|w_2|\} \leq \epsilon < 1$. Let $A_P := \{(w_1, \dots, w_n) \in X_P \mid w_1 = w_2 = 0\}$. Let $q : X_P \rightarrow A_P$ be the projection. Because $\dim_{\mathbb{R}} Z'_1 = 2n - 5$, we have a closed subanalytic subset $Z''_1 \subset A_P$ such that $q^{-1}(Q) \cap (Z'_1 \cap X_P) = \emptyset$ for any $Q \in A_P \setminus Z'_1$. Hence, by using [41, Theorem 7.4], we obtain that $V_2^{DM}|_{X_P \setminus Z'_1}$ is uniquely extended to a reflexive $\mathcal{O}_{X_P \setminus Z'_1}$ -module. It implies that (V_2, ∇) is extended to a meromorphic flat connection (V_3, ∇) on $(X \setminus Z_2, H \setminus Z_2)$.

By an induction with a similar argument, we obtain that (V_3, ∇) is extended to a meromorphic flat connection (V, ∇) on (X, H) .

Lemma 9.4 *Let $\varphi : \Delta \rightarrow X$ such that $\varphi(0) \in H$ and $\varphi(\Delta \setminus \{0\}) \subset X \setminus H$. Then, $\varphi^{-1}K$ and $\text{DR}^E \varphi^*V$ are naturally isomorphic.*

Proof If $\varphi(0) \in H \setminus Z$, the claim holds by the construction of (V, ∇) . For a general φ , we can take $\psi : \Delta^2 \rightarrow X$ such that (i) $\varphi(w_1, 0) = \varphi(w_1)$, (ii) for general w_2 , we have $\varphi(0, w_2) \in H \setminus Z$. By applying the result in the surface case (Proposition 9.1), we have a meromorphic flat bundle (V_{100}, ∇) on $(\Delta^2, \{w_1 = 0\})$ such that $\text{DR}^E(V_{100}) \simeq E\psi^{-1}(K)$. We have $V_{100} \simeq \psi^*V$ outside of $Z_{100} \subset \{w_1 = 0\}$ with $\dim Z_{100} = 0$. Hence, we have $V_{100} \simeq \psi^*(V)$ on Δ . It implies the claim of the lemma. \blacksquare

Then, we obtain a global isomorphism $\text{DR}^E(V)[-n] \simeq K$ from Proposition 3.31 in this case.

9.2.2 Normal crossing case

Let us consider the case where $H = \bigcup H_j$ is normal crossing. Set $H^{[2]} = \bigcup_{i \neq j} (H_i \cap H_j)$. We have already extended (V_0, ∇) on $X \setminus H$ to (V_1, ∇) on $X \setminus H^{[2]}$. The Deligne-Malgrange lattice of V_1 is a coherent reflexive \mathcal{O} -module on $X \setminus H^{[2]}$. By using the result in the surface case (Proposition 9.1), and by using the extension theorem of Siu [41, Theorem 7.4], we obtain that (V_0, ∇) is extended to a meromorphic flat sheaf on (X, H) . Then, the claim of Theorem 9.3 follows from Proposition 3.31 and the next lemma.

Lemma 9.5 *Let $\varphi : \Delta \rightarrow X$ be any holomorphic map such that $\varphi(0) \in H$ and $\varphi(\Delta \setminus \{0\}) \subset X \setminus H$. Then, $E\varphi^{-1}(K)$ and $DR^E \varphi^* V[-1]$ are naturally isomorphic.*

Proof We take a projective birational morphism $\rho : X' \rightarrow X$ such that (i) $H' = \rho^{-1}(H)$ is normal crossing, (ii) $X' \setminus H' \simeq X \setminus H$, (iii) $\varphi'(0)$ is contained in the smooth part of the exceptional divisor of ρ for the strict transform $\varphi' : \Delta \rightarrow X'$ of φ . Applying the consideration in the beginning of this subsection to $E\rho^{-1}(K)$, we obtain a meromorphic flat bundle (V', ∇) on (X', H') as the extension of $\rho^*(V_0, \nabla)$ corresponding to $E\rho^{-1}K$. Then, we have $DR^E(\varphi')^*(V')[-1] \simeq E(\varphi')^{-1}\rho^{-1}(K)$ by the result in the smooth case §9.2.1. Because $\rho_* V'$ is naturally isomorphic to V as meromorphic flat connections, we have $V' \simeq \rho^* V$. Then, the claim of Lemma 9.5 follows. The proof of Theorem 9.3 is also completed. \blacksquare

10 Holonomic \mathcal{D} -modules and enhanced ind-sheaves

10.1 Statement

Let X be a complex manifold. Let $E_\Delta^b(IC_X) \subset E_{\mathbb{R}-c}^b(IC_X)$ denote the full subcategory of objects K with the following property.

- Let $\varphi : \Delta \rightarrow X$ be any holomorphic map. Then, $E\varphi^{-1}(K)$ comes from a cohomologically holonomic \mathcal{D}_Δ -complex, i.e., we have an object $M \in D_{\text{hol}}^b(\mathcal{D}_\Delta)$ and an isomorphism $E\varphi^{-1}(K) \simeq DR_\Delta^E(M)$.

The functor $\text{Sol}^E : D_{\text{hol}}^b(\mathcal{D}_X)^{\text{op}} \rightarrow E_{\mathbb{R}-c}^b(IC_X)$ factors through $E_\Delta^b(IC_X)$. It is fully faithful according to [3]. Let us prove the following in the rest of this paper.

Theorem 10.1 $\text{Sol}^E : D_{\text{hol}}^b(\mathcal{D}_X)^{\text{op}} \rightarrow E_\Delta^b(IC_X)$ is an equivalence.

10.2 Construction of quasi-inverse

Let $K \in E^b(IC_X)$. We set $\Upsilon^E(K) := \mathcal{H}om^E(K, \mathcal{O}^E)$ in $D^b(\mathcal{D}_X)$. Let us observe that we have a natural morphism $\Phi_K : K \rightarrow \text{Sol}^E(\Upsilon^E(K))$. Let $\pi : \mathbb{R} \times X \rightarrow X$ denote the projection. Let $j_X : \mathbb{R}_\infty \times X \rightarrow \mathbb{R} \times X$ denote the natural morphism of bordered spaces. We set $\tilde{L}^E(G) := Rj_{X!} L^E G$ and $\tilde{R}^E(G) := Rj_{X*} R^E G$ in $D^b(IC_{\mathbb{R} \times X})$. According to [3, Definition 4.13], we naturally have

$$\Upsilon^E(K) \simeq R\pi_* R\mathcal{H}om_{IC_{\mathbb{R} \times X}}(\tilde{L}^E K, \tilde{R}^E \mathcal{O}^E).$$

We have the following natural morphism

$$R\mathcal{H}om_{\beta\pi^{-1}\mathcal{D}_X}(\beta R\mathcal{H}om_{IC_{\mathbb{R} \times X}}(\tilde{L}^E K, \tilde{R}^E \mathcal{O}^E), \tilde{R}^E \mathcal{O}^E) \rightarrow R\mathcal{H}om_{\beta\pi^{-1}\mathcal{D}_X}(\beta\pi^{-1} R\pi_* R\mathcal{H}om_{IC_{\mathbb{R} \times X}}(\tilde{L}^E K, \tilde{R}^E \mathcal{O}^E), \tilde{R}^E \mathcal{O}^E). \quad (39)$$

We have the natural quotient functor $Q : D^b(IC_{\mathbb{R} \times X}) \rightarrow E^b(IC_X)$. We have the following natural isomorphism:

$$\begin{aligned} \text{Sol}^E(\Upsilon^E(K)) &\simeq R\mathcal{H}om_{\beta\pi^{-1}\mathcal{D}_X}(\beta\pi^{-1} R\pi_* R\mathcal{H}om_{IC_{\mathbb{R} \times X}}(\tilde{L}^E K, \tilde{R}^E \mathcal{O}^E), R^E \mathcal{O}^E) \\ &\simeq QR\mathcal{H}om_{\beta\pi^{-1}\mathcal{D}_X}(\beta\pi^{-1} R\pi_* R\mathcal{H}om_{IC_{\mathbb{R} \times X}}(\tilde{L}^E K, \tilde{R}^E \mathcal{O}^E), \tilde{R}^E \mathcal{O}^E). \end{aligned} \quad (40)$$

By (39) and (40), we have the following natural morphism:

$$QRHom_{\beta\pi^{-1}\mathcal{D}_X}(\beta RHom_{IC_{\mathbb{R} \times X}}(\tilde{L}^E K, \tilde{R}^E \mathcal{O}^E), \tilde{R}^E \mathcal{O}^E) \longrightarrow \text{Sol}^E(\Upsilon^E(K)). \quad (41)$$

We have the following natural isomorphisms:

$$\begin{aligned} RHom_{IC_{\mathbb{R} \times X}}(\tilde{L}^E K, QRHom_{\beta\pi^{-1}\mathcal{D}_X}(\beta RHom_{IC_{\mathbb{R} \times X}}(\tilde{L}^E K, \tilde{R}^E \mathcal{O}^E), \tilde{R}^E \mathcal{O}^E)) &\simeq \\ RHom_{I(\beta\pi^{-1}\mathcal{D}_X)}(\beta RHom_{IC_{\mathbb{R} \times X}}(\tilde{L}^E K, \tilde{R}^E \mathcal{O}^E), QRHom_{IC_{\mathbb{R} \times X}}(\tilde{L}^E K, \tilde{R}^E \mathcal{O}^E)) &\simeq \\ RHom_{I(\pi^{-1}\mathcal{D}_X)}(RHom_{IC_{\mathbb{R} \times X}}(\tilde{L}^E K, \tilde{R}^E \mathcal{O}^E), RHom_{IC_{\mathbb{R} \times X}}(\tilde{L}^E K, \tilde{R}^E \mathcal{O}^E)). \end{aligned} \quad (42)$$

Here, we obtain the first isomorphism from more general isomorphisms (44) below and [15, Proposition 5.4.11], and we obtain the second isomorphism by [15, Theorem 5.6.2]. By (41) and (42), we have the morphism $\Phi_K : K \longrightarrow \text{Sol}^E(\Upsilon^E(K))$ in $E^b(IC_X)$ corresponding to the identity:

$$\text{id} \in RHom_{I(\pi^{-1}\mathcal{D}_X)}(RHom_{IC_{\mathbb{R} \times X}}(\tilde{L}^E K, \tilde{R}^E \mathcal{O}^E), RHom_{IC_{\mathbb{R} \times X}}(\tilde{L}^E K, \tilde{R}^E \mathcal{O}^E)).$$

Lemma 10.2 *Let $\psi : K_1 \longrightarrow K_2$ be a morphism in $E^b(IC_X)$. Then, the following induced diagram is commutative:*

$$\begin{array}{ccc} K_1 & \xrightarrow{a_1} & \text{Sol}^E(\Upsilon^E(K_1)) \\ \psi \downarrow & & \downarrow \psi_* \\ K_2 & \xrightarrow{a_2} & \text{Sol}^E(\Upsilon^E(K_2)). \end{array}$$

Here, the right vertical arrow is the morphism induced by ψ .

Proof It is easy to check that both $\psi_* \circ a_1$ and $a_2 \circ \psi$ correspond to

$$\psi^* \in RHom_{\pi^{-1}\mathcal{D}_X}(RHom_{IC_{\mathbb{R} \times X}}(\tilde{L}^E K_2, \tilde{R}^E \mathcal{O}^E), RHom_{IC_{\mathbb{R} \times X}}(\tilde{L}^E K_1, \tilde{R}^E \mathcal{O}^E)).$$

■

10.2.1 Appendix

Let Y be a good topological space. Let \mathcal{A} be a ring in IC_M . We have the following natural isomorphism for any $\mathcal{N}, \mathcal{M} \in I(\mathcal{A})$ and $M \in I(\mathbb{C}_Y)$:

$$Thom_{\mathcal{A}}(\mathcal{N} \otimes_{IC_Y} M, \mathcal{M}) \simeq Thom_{\mathcal{A}}(\mathcal{N}, Thom_{IC_Y}(M, \mathcal{M})). \quad (43)$$

Indeed, in the case $\mathcal{A} = \mathbb{C}_X$, it follows from [15, Corollary 4.2.9]. Then, we can easily obtain (43) for general \mathcal{A} from the definition of $Thom_{\mathcal{A}}(\cdot, \cdot)$ [15, Definition 5.4.9].

We have the following natural isomorphism in $D^+(IC_Y)$ for any $\mathcal{N} \in D^-(IA)$, $\mathcal{M} \in D^+(IA)$ and $M \in D^-(IC_Y)$:

$$RThom_{\mathcal{A}}(\mathcal{N} \otimes_{IC_Y} M, \mathcal{M}) \simeq RThom_{\mathcal{A}}(\mathcal{N}, RThom_{IC_Y}(M, \mathcal{M})). \quad (44)$$

It follows from the construction of the derived functor $RThom_{\mathcal{A}}(\cdot, \cdot)$ in [15, §5.1, §5.4].

10.3 Preliminary

10.3.1 \mathbb{R} -linear subspaces

Let V be a finite dimensional \mathbb{C} -vector space. Let H be an \mathbb{R} -subspace of V . For any real number a , we set $[a]_+ := \min\{\ell \in \mathbb{Z} \mid \ell \geq a\}$ and $[a]_- := \max\{\ell \in \mathbb{Z} \mid \ell \leq a\}$.

Lemma 10.3 *We have a \mathbb{C} -subspace L of V such that $L \cap H = 0$ and $\dim_{\mathbb{C}} L = \dim_{\mathbb{C}} V - [\dim_{\mathbb{R}} H/2]_+$.*

Proof We set $H' := H \cap \sqrt{-1}H$ which is a \mathbb{C} -subspace of V . We take an \mathbb{R} -subspace $H'' \subset H$ such that $H = H' \oplus H''$. We have $[\dim_{\mathbb{R}} H/2]_+ = \dim_{\mathbb{C}} H' + [\dim_{\mathbb{R}} H''/2]_+$. The \mathbb{C} -subspace $H + \sqrt{-1}H$ is equal to $H' \oplus (H'' \oplus \sqrt{-1}H'')$. Hence, we have $\dim_{\mathbb{C}}(H + \sqrt{-1}H) = \dim_{\mathbb{C}} H' + \dim_{\mathbb{R}} H''$.

In general, we can easily construct a \mathbb{C} -subspace L_1 of \mathbb{C}^ℓ such that $L_1 \cap \mathbb{R}^\ell = 0$ and $\dim_{\mathbb{C}} L_1 = [\ell/2]_-$. Hence, we can easily construct a \mathbb{C} -subspace $L \subset V$ such that $V \cap H = 0$ and

$$\dim_{\mathbb{C}} L = \dim_{\mathbb{C}} V - \dim_{\mathbb{C}}(H + \sqrt{-1}H) + [\dim_{\mathbb{R}} H''/2]_- = \dim_{\mathbb{C}} V - [\dim_{\mathbb{R}} H/2]_+.$$

Thus, we obtain the claim of the lemma. ■

Lemma 10.4 *Suppose that H is not a \mathbb{C} -subspace of V , and that $\dim_{\mathbb{R}} H = 2 \dim_{\mathbb{C}} V - 2$. Then, for any $u \in V/H$, we have $v \in H$ such that $\sqrt{-1}v$ is mapped to u via the projection $V \rightarrow V/H$.*

Proof Set $H' := H \cap \sqrt{-1}H$. Set $V_1 := V/H'$ and $H_1 := H/H'$. Because H' is not a \mathbb{C} -subspace of V , we have $\dim_{\mathbb{C}} H' = \dim_{\mathbb{C}} V - 2$, and hence $\dim_{\mathbb{C}} V_1 = \dim_{\mathbb{R}} H_1 = 2$. It implies that $V_1 = H_1 \oplus \sqrt{-1}H_1$. Then, the claim of the lemma is clear. ■

Lemma 10.5 *Suppose that $\dim_{\mathbb{R}} H = 2m - 1$ for a positive integer m . Let $\mathcal{U}(V, H, m) \subset \text{Hom}(V, \mathbb{C}^m)$ be the set of \mathbb{C} -linear maps $f : V \rightarrow \mathbb{C}^m$ such that $f|_H$ is injective. Then, $\mathcal{U}(V, H, m)$ is a non-empty open subset.*

Proof It is enough to check that $\mathcal{U}(V, H, m)$ is non-empty. By Lemma 10.3, we have a \mathbb{C} -subspace $L \subset V$ such that $L \cap H = 0$ and $\dim_{\mathbb{C}} L = \dim_{\mathbb{C}} V - m$. Then, we have only to take the composition of the projection $V \rightarrow V/L$ and a \mathbb{C} -isomorphism $V/L \simeq \mathbb{C}^m$. ■

10.3.2 Subanalytic subsets which are generically complex analytic

Let X be a complex manifold. Let $Z \subset X$ be a purely k -dimensional closed subanalytic subset. Let $Z_0 \subset Z$ be a closed subanalytic subset with $\dim_{\mathbb{R}} Z_0 < k$. Suppose that $Z \setminus Z_0$ is a complex submanifold of X . In particular, k is even.

By a generalization of the theorem of Remmert-Stein [36] due to Shiffman [39], the following holds.

Proposition 10.6 *If $\dim_{\mathbb{R}} Z_0 < k - 1$, then Z is a complex analytic subvariety of X .* ■

Let us study a description of Z around a general point of Z_0 in the case $\dim_{\mathbb{R}} Z_0 = k - 1$.

Proposition 10.7 *Suppose $\dim_{\mathbb{R}} Z_0 = k - 1$. Then, there exists a closed subanalytic subset $Z_1 \subset Z_0$ with $\dim_{\mathbb{R}} Z_1 \leq k - 2$ such that the following holds.*

- Z_1 contains the singular locus of Z_0 .
- For any $P \in Z_0 \setminus Z_1$, we have a neighbourhood X_P of P in X with a real analytic function $f_P : X_P \rightarrow \mathbb{R}$ and a closed $(k/2)$ -dimensional complex submanifold $\tilde{Z}_P \subset X_P$ such that (i) $X_P \cap Z_0 = \tilde{Z}_P \cap f^{-1}(0)$, (ii) $df|_{\tilde{Z}_P}$ is nowhere vanishing, (iii) $X_P \cap Z$ is equal to \tilde{Z}_P or $\tilde{Z}_P \cap f^{-1}(\mathbb{R}_{\geq 0})$.

Proof We may assume that X is an open subset of \mathbb{C}^n . Let Q be any point of Z_0 . We take a relatively compact neighbourhood X_Q of Q in X . Set $Z_Q := Z \cap X_Q$ and $Z_{0,Q} := Z_0 \cap X_Q$. Let $Z_{0,Q}^{\text{sm}}$ denote the set of the smooth points of $Z_{0,Q}$. It is decomposed into the union of connected components \mathcal{C}_i ($i = 1, \dots, \ell_1(Q)$). We also have the decomposition of $Z_Q \setminus Z_{0,Q}$ into the connected component \mathcal{D}_j ($j = 1, \dots, \ell_2(Q)$). We take points $R(\mathcal{C}_i) \in \mathcal{C}_i$ ($i = 1, \dots, \ell_1(Q)$) and $R(\mathcal{D}_j) \in \mathcal{D}_j$ ($j = 1, \dots, \ell_2(Q)$).

By Lemma 10.5, for $i = 1, \dots, \ell_1(Q)$, we can take a \mathbb{C} -linear map $\phi_i : \mathbb{C}^n \rightarrow \mathbb{C}^{k/2}$ such that (i) the restriction of ϕ_i to $T_{R(\mathcal{C}_i)}\mathcal{C}_i$ is injective, (ii) the restriction of ϕ_i to $T_{R(\mathcal{D}_j)}\mathcal{D}_j$ are isomorphisms for any j . Hence, we have a closed subanalytic subset $W_{1,i} \subset Z_{0,Q}$ with $\dim_{\mathbb{R}} W_{1,i} \leq k - 2$, and $W_{2,i} \subset Z_Q$ with $\dim_{\mathbb{R}} W_{2,i} \leq k - 1$, such that (i) the restriction of ϕ_i to $\mathcal{C}_i \setminus W_{1,i}$ is an immersion, (ii) the restriction of ϕ_i to $\mathcal{D}_j \setminus W_{2,i}$ are immersions for any j . We obtain the k -dimensional subanalytic subset $\phi_i(Z_Q) \subset \mathbb{C}^{k/2} \simeq \mathbb{R}^k$. We have a subanalytic closed subset $W_{3,i} \subset \phi_i(Z_Q)$ such that (i) $\dim_{\mathbb{R}} W_{3,i} \leq k - 1$, (ii) each connected component of $\phi_i(Z_Q) \setminus W_{3,i}$ is simply connected, (iii) $W_{3,i}$ contains $\partial\phi_i(Z_Q)$ and $\phi_i(Z_{0,Q})$, (iv) the restriction of ϕ_i to $Z_Q \setminus \phi_i^{-1}(W_{3,i}) \rightarrow \mathbb{C}^{k/2}$ is a local diffeomorphism.

We take a \mathbb{C} -isomorphism $\mathbb{C}^n \simeq \mathbb{C}^{k/2} \oplus H_i$ such that ϕ_i is the projection onto the first component. On each connected component \mathcal{N} of $\phi_i(Z_Q) \setminus W_{3,i}$, we have H_i -valued subanalytic functions $h_{\mathcal{N},p}$ ($p \in \Lambda(i, \mathcal{N})$) on $(\mathcal{N}, \mathbb{C}^{k/2})$ such that $\phi_i^{-1}(\mathcal{N}) \cap Z_Q$ is expressed as the union of the graph $\Gamma(i, h_{\mathcal{N},p})$. By Lemma 2.12, we have a closed subanalytic subset $W_{4,i} \subset W_{3,i}$ such that (i) $\dim_{\mathbb{R}} W_{4,i} \leq k-2$, (ii) for each connected component \mathcal{N} of $\phi_i(Z_Q) \setminus W_{3,i}$, the singular locus of $\partial\mathcal{N}$ is contained in $W_{4,i}$, and $h_{\mathcal{N},p}$ are ramified analytic around any point of $\partial\mathcal{N} \setminus W_{4,i}$. Because $h_{\mathcal{N},p}$ are holomorphic on \mathcal{N} , we have a neighbourhood \mathcal{M}_R for each point $R \in \partial\mathcal{N} \setminus W_{4,i}$ such that $h_{\mathcal{N},p}|_{\mathcal{M}_R \cap \mathcal{N}}$ are extended to a holomorphic function on \mathcal{M}_R .

Set $W_1 := \bigcup_i W_{1,i}$. We have $\dim_{\mathbb{R}} W_1 \leq k-2$. We also have the following.

Lemma 10.8 *We have $\dim_{\mathbb{R}} \phi_i^{-1}(W_{4,i}) \leq k-2$.*

Proof We have $\dim_{\mathbb{R}}(\phi_i^{-1}(W_{4,i}) \cap W_1) \leq \dim_{\mathbb{R}} W_1 \leq k-2$. Because the restriction of ϕ_i to $\mathcal{C}_i \setminus W_1$ is immersion, we have $\dim_{\mathbb{R}}(\phi_i^{-1}(W_{4,i}) \setminus W_1) \leq \dim_{\mathbb{R}} W_{4,i} \leq k-2$. Thus, we obtain the claim of the lemma. \blacksquare

Let $\bar{\Gamma}(i, h_{\mathcal{N},p})$ denote the closure of $\Gamma(i, h_{\mathcal{N},p})$ in Z_Q . We set

$$E((i, \mathcal{N}, p), (i', \mathcal{N}', p')) := \bar{\Gamma}(i, h_{\mathcal{N},p}) \cap \bar{\Gamma}(i', h_{\mathcal{N}',p'}).$$

Let $E((i, \mathcal{N}, p), (i', \mathcal{N}', p'))_q^{\text{sm}}$ denote the set of q -dimensional smooth points of $E((i, \mathcal{N}, p), (i', \mathcal{N}', p'))$.

Lemma 10.9 *Let P be any point of $E((i, \mathcal{N}, p), (i', \mathcal{N}', p'))_{k-1}^{\text{sm}} \cap Z_{0,Q}^{\text{sm}} \setminus (W_1 \cup \bigcup_i \phi_i^{-1}(W_{4,i}))$. Then, the intersection of $\bar{\Gamma}(i, h_{\mathcal{N},p}) \cup \bar{\Gamma}(i', h_{\mathcal{N}',p'})$ and a neighbourhood of P is a complex submanifold.*

Proof We take a small neighbourhood X_P of P in X . By construction, we have a complex submanifold $Y(i, \mathcal{N}, p)$ and a real analytic map $f : X_P \rightarrow \mathbb{R}$ such that $\bar{\Gamma}(i, h_{\mathcal{N},p}) = Y(i, \mathcal{N}, p) \cap f^{-1}(\mathbb{R}_{\geq 0})$. We also have a complex submanifold $Y(i', \mathcal{N}', p')$ and a real analytic map $f' : X_P \rightarrow \mathbb{R}$ such that $\bar{\Gamma}(i', h_{\mathcal{N}',p'}) = Y(i', \mathcal{N}', p') \cap (f')^{-1}(\mathbb{R}_{\geq 0})$. Because $\dim_{\mathbb{R}}(Y(i, \mathcal{N}, p) \cap Y(i', \mathcal{N}', p')) \geq k-1$, we have $Y(i, \mathcal{N}, p) = Y(i', \mathcal{N}', p')$. Then, the claim of the lemma is clear. \blacksquare

Let $\text{Sing } \overline{E((i, \mathcal{N}, p), (i', \mathcal{N}', p'))_{k-1}^{\text{sm}}}$ denote the singular locus of $\overline{E((i, \mathcal{N}, p), (i', \mathcal{N}', p'))_{k-1}^{\text{sm}}}$. We set

$$W_5 := \bigcup_{(i, \mathcal{N}, p), (i', \mathcal{N}', p')} \left(\bigcup_{q \leq k-2} \overline{E((i, \mathcal{N}, p), (i', \mathcal{N}', p'))_q^{\text{sm}}} \cup \text{Sing } \overline{E((i, \mathcal{N}, p), (i', \mathcal{N}', p'))_{k-1}^{\text{sm}}} \right).$$

We have $\dim_{\mathbb{R}}(\phi_i^{-1}(W_{3,i}) \cap Z_Q) \leq k-1$. Let $W_{6,i}$ denote the closure of $\phi_i^{-1}(W_{3,i}) \cap (Z_Q \setminus Z_{0,Q})$ in Z_Q . Let W_7 denote the union of the singular locus of $Z_{0,Q}$ and $W_1 \cup \bigcup_i \phi_i^{-1}(W_{4,i}) \cup \bigcup_i (W_{6,i} \cap Z_{0,Q}) \cup W_5$. We have $\dim_{\mathbb{R}} W_7 \leq k-2$.

Take any point $P \in Z_{0,Q} \setminus W_7$. Take a small neighbourhood X_P of P in X . We have at least one (i, \mathcal{N}, p) such that $P \in \bar{\Gamma}(i, h_{\mathcal{N},p})$. If we have another (i', \mathcal{N}', p') such that $P \in \bar{\Gamma}(i', h_{\mathcal{N}',p'})$, then $\bar{\Gamma}(i, h_{\mathcal{N},p}) \cup \bar{\Gamma}(i', h_{\mathcal{N}',p'})$ gives a complex submanifold. Thus, we obtain the claim of Proposition 10.7. \blacksquare

Let $A \subset X$ be a closed complex subvariety. Let $B \subset A$ be a closed subset such that (i) B is a purely q -dimensional subanalytic subset of X , (ii) $B \setminus A$ is a complex subvariety of $X \setminus A$.

Corollary 10.10 *Under the assumption, B is a complex subvariety of X .*

Proof Let $\text{Sing}(B)$ denote the singular locus of B . It is enough to prove that $\dim_{\mathbb{R}} \text{Sing}(B) < 2q-1$ by Proposition 10.6. By the assumption, we have $\dim_{\mathbb{R}}(\text{Sing}(B) \setminus A) < 2q-1$. Hence, it is enough to prove $\dim_{\mathbb{R}}(\text{Sing}(B) \cap A) < 2q-1$.

We assume that $\dim_{\mathbb{R}}(\text{Sing}(B) \cap A) = 2q-1$, and we shall derive a contradiction. We have a closed subanalytic subset $W \subset \text{Sing}(B) \cap A$ with $\dim_{\mathbb{R}} W < 2q-1$ such that the following holds for any $P \in (\text{Sing}(B) \cap A) \setminus W$.

- P is a smooth point of $\text{Sing}(B)$.
- Take a small neighbourhood X_P of P in X . Then, we have a closed complex submanifold \tilde{B}_P of X_P and a real analytic function $f_P : \tilde{B}_P \rightarrow \mathbb{R}$ whose exterior derivative is nowhere vanishing, such that B is $\tilde{B}_P \cap f_P^{-1}(\mathbb{R}_{\geq 0})$.

Because $\tilde{B}_P \not\subset A$, we have $\dim_{\mathbb{R}}(\tilde{B}_P \cap A) \leq \dim_{\mathbb{R}} \tilde{B}_P - 2 = 2q-2$. It contradicts with $\dim_{\mathbb{R}}(\text{Sing}(B) \cap A) = 2q-1$. Thus, we obtain Corollary 10.10. \blacksquare

10.3.3 \mathbb{R} -constructible sheaves

Let X be any complex manifold. Let $D_{\Delta}^b(\mathbb{C}_X) \subset D_{\mathbb{R}-c}^b(\mathbb{C}_X)$ denote the full subcategory of objects K with the following property.

- Let $\varphi : \Delta \rightarrow X$ be any holomorphic map. Then, $\varphi^{-1}(K)$ is cohomologically \mathbb{C} -constructible.

Lemma 10.11 $D_{\Delta}^b(\mathbb{C}_X)$ is a triangulated subcategory of $D_{\mathbb{R}-c}^b(\mathbb{C}_X)$.

Proof Let $K_1 \rightarrow K_2 \rightarrow K_3 \rightarrow K_1[1]$ be a distinguished triangle in $D_{\mathbb{R}-c}^b(\mathbb{C}_X)$ such that K_i ($i = 1, 2$) are objects in $D_{\Delta}^b(\mathbb{C}_X)$. Let $\varphi : \Delta \rightarrow X$ be any holomorphic map. We set $K'_i := E\varphi^{-1}(K_i)$. We obtain a distinguished triangle $K'_1 \rightarrow K'_2 \rightarrow K'_3 \rightarrow K'_1[1]$ in $D_{\mathbb{R}-c}^b(\mathbb{C}_{\Delta})$. For $i = 1, 2$, K'_i are cohomologically \mathbb{C} -constructible. Then, K'_3 is also cohomologically \mathbb{C} -constructible. Hence, we obtain $K_3 \in D_{\Delta}^b(\mathbb{C}_X)$. \blacksquare

Lemma 10.12 An object $K \in D_{\mathbb{R}-c}^b(\mathbb{C}_X)$ is contained in $D_{\Delta}^b(\mathbb{C}_X)$ if and only if $\mathcal{H}^j(K)$ ($j \in \mathbb{Z}$) are objects in $D_{\Delta}^b(\mathbb{C}_X)$. Here, $\mathcal{H}^j(K)$ denotes the j -th cohomology sheaf.

Proof Let us prove the “if” part. For any K , we have $\ell(K) := \max\{n \mid \mathcal{H}^n(K) \neq 0\} - \min\{n \mid \mathcal{H}^n(K) \neq 0\}$. We use an induction on $\ell(K)$. Let K be an object in $D_{\mathbb{R}-c}^b(\mathbb{C}_X)$ such that $\mathcal{H}^j(K)$ are objects in $D_{\Delta}^b(\mathbb{C}_X)$. If $\ell(K) = 0$, we clearly have $K \in D_{\Delta}^b(\mathbb{C}_X)$. We have n such that $K = \tau^{\leq n}(K)$ and $\mathcal{H}^n(K) \neq 0$. We have the distinguished triangle $K \rightarrow \mathcal{H}^n(K) \rightarrow \tau^{\leq n-1}(K)[1] \rightarrow K[1]$ in $D_{\mathbb{R}-c}^b(\mathbb{C}_{\Delta})$. By the assumption of the induction, $\tau^{\leq n-1}(K)[1]$ is an object in $D_{\Delta}^b(\mathbb{C}_{\Delta})$. Because $D_{\Delta}^b(\mathbb{C}_{\Delta})$ is triangulated, we obtain that $K \in D_{\Delta}^b(\mathbb{C}_{\Delta})$.

Let us prove the “only if” part. Let K be an object in $D_{\Delta}^b(\mathbb{C}_X)$. Let $\varphi : \Delta \rightarrow X$ be any holomorphic map. The cohomology sheaves $\mathcal{H}^j \varphi^{-1}(K)$ on Δ are \mathbb{C} -constructible. Because $\mathcal{H}^j \varphi^{-1}(K) \simeq \varphi^{-1} \mathcal{H}^j(K)$, we have $\mathcal{H}^j \in D_{\Delta}^b(\mathbb{C}_X)$. \blacksquare

Let $Y \subset X$ be a real analytic submanifold with $\dim_{\mathbb{R}} Y \leq \dim_{\mathbb{R}} X - 2$. Let K be an \mathbb{R} -constructible sheaf on X such that $K|_{X \setminus Y}$ and $K|_Y$ are local systems on $X \setminus Y$ and Y , respectively.

Lemma 10.13 Suppose that $K \in D_{\Delta}^b(\mathbb{C}_X)$. Then, for any $P \in Y$, we have a neighbourhood X_P of P in X such that one of the following holds; (i) $K|_{X_P}$ is a local system on X_P , (ii) $Y \cap X_P$ is a complex submanifold of X_P .

Proof We shall shrink X and Y around P . We may assume that Y is simply connected, and $X = Y \times]0, 1[^a$ as a C^{∞} -manifold, where $a = \dim_{\mathbb{R}} X - \dim_{\mathbb{R}} Y$.

Let us consider the case $\dim_{\mathbb{R}} Y < \dim_{\mathbb{R}} X - 2$. We have a local system L on X with an isomorphism $\kappa_0 : K|_{X \setminus Y} \simeq L|_{X \setminus Y}$. We have a natural morphism of \mathbb{R} -constructible sheaves $\kappa : K \rightarrow L$ such that $\kappa|_{X \setminus Y} = \kappa_0$. Note that $L \in D_{\Delta}^b(\mathbb{C}_X)$. By Lemma 10.12, $\text{Cok}(\varphi)$ and $\text{Ker}(\varphi)$ are objects in $D_{\Delta}^b(\mathbb{C}_X)$. Note that $\text{Cok}(\varphi)$ and $\text{Ker}(\varphi)$ are local systems on Y . Hence, we have one of the following; (i) $\text{Cok}(\varphi) = \text{Ker}(\varphi) = 0$, i.e., φ is an isomorphism, (ii) Y is a complex submanifold.

Let us consider the case $\dim_{\mathbb{R}} Y = \dim_{\mathbb{R}} X - 2$. We set $L_0 := K|_{X \setminus Y}$. We have the automorphism $F : L_0 \rightarrow L_0$ obtained as the monodromy along a loop along $Y \times \{0\}$ with counter clock-wise direction. We have the generalized eigen decomposition $L_0 = \bigoplus_{\alpha \in \mathbb{C}} \mathbb{E}_{\alpha} L_0$, which is compatible with the action of F and the restriction of $F - \alpha \text{id}$ to $\mathbb{E}_{\alpha} L_0$ is nilpotent for each α .

Let $\iota : X \setminus Y \rightarrow X$ denote the inclusion. We have the decomposition $K = \bigoplus_{\alpha \in \mathbb{C}} K_{\alpha}$, where $K_{\alpha} \simeq \iota_{*} \mathbb{E}_{\alpha} L_0$ ($\alpha \neq 1$), and $K|_{X \setminus Y} \simeq \mathbb{E}_1 L_0$. Let us prove that Y is a complex submanifold of X in the case where $\mathbb{E}_{\alpha} L_0 \neq 0$ for one of $\alpha \neq 1$. Suppose that Y is not a complex submanifold. Then, we have a holomorphic map $\varphi : \Delta \rightarrow X$ such that $\varphi^{-1}(Y) = \{t \in \mathbb{R}\} \cap \Delta$. Then, $\varphi^{-1}(K)$ is not \mathbb{C} -constructible on Δ , which contradicts with our assumption $K \in D_{\Delta}^b(\mathbb{C}_X)$. Hence, we obtain that Y is a complex submanifold. Let us consider the case where $\mathbb{E}_{\alpha} L_0 = 0$ for any $\alpha \neq 1$. We shall prove the claim of the lemma in this case by using an induction on L_0 . If $\text{rank } L_0 = 0$, K is a local system on Y , and hence the claim of the lemma is clear. If $\text{rank } L_0 > 0$, we have an epimorphism $L_0 \rightarrow \mathbb{C}_{X \setminus Y}$ because the monodromy is unipotent. We have the induced morphism $\iota_{*} L_0 \rightarrow \mathbb{C}_X$. Let $\rho : K \rightarrow \mathbb{C}_X$ denote the composition of the morphisms $K \rightarrow \iota_{*} L_0 \xrightarrow{\iota_{*} \rho} \mathbb{C}_X$. Note that $\mathbb{C}_X \in D_{\Delta}^b(\mathbb{C}_X)$. Hence, by Lemma 10.12, we obtain that $\text{Ker}(\rho)$ and $\text{Cok}(\rho)$ are objects in $D_{\Delta}^b(\mathbb{C}_X)$. Note that $\text{Cok}(\rho)$ is a local

system on Y . Hence, if $\text{Cok}(\rho) \neq 0$, we obtain that Y is a complex submanifold of X . Suppose that $\text{Cok}(\rho) = 0$. Note that $\text{Ker}(\rho)|_{X \setminus Y}$ and $\text{Ker}(\rho)|_Y$ are local systems. Moreover, the monodromy of $\text{Ker}(\rho)|_{X \setminus Y}$ is unipotent, and $\text{rank } \text{Ker}(\rho)|_{X \setminus Y} < \text{rank } L_0$. Hence, by using the assumption of the induction, we obtain the claim of the lemma in this case. Thus, the proof of Lemma 10.13 is finished. \blacksquare

Let Y and X be as above. Let f be a real analytic function $Y \rightarrow \mathbb{R}$ such that df is nowhere vanishing. Set $Y_{\geq 0} := f^{-1}(\mathbb{R}_{\geq 0})$ and $Y_0 := f^{-1}(0)$. Let K be an \mathbb{R} -constructible sheaf on X such that $K|_{X \setminus Y_{\geq 0}}$, $K|_{Y_{\geq 0} \setminus Y_0}$ and $K|_{Y_0}$ are local systems on $X \setminus Y_{\geq 0}$, $Y_{\geq 0} \setminus Y_0$ and Y_0 , respectively.

Lemma 10.14 *Suppose that $K \in D_{\Delta}^b(\mathbb{C}_X)$. Then, for any $P \in Y_0$, we have a small neighbourhood X_P of P in X such that $K|_{X_P}$ is a local system on X_P .*

Proof Let P be any point of Y_0 . We take a small neighbourhood X_P . Because $X_P \setminus Y_{\geq 0}$ is simply connected, we have a local system L_P on X_P with an isomorphism $L_P \simeq K|_{X_P \setminus Y_{\geq 0}}$. Then, we have a natural morphism $\varphi : K|_{X_P} \rightarrow L_P$. We set $K_1 := \text{Ker } \varphi$ and $K_2 := \text{Cok } \varphi$. Then, we have $K_i|_{X \setminus Y_{\geq 0}} = 0$, and $K_i|_{Y_{\geq 0} \setminus Y_0}$ and $K_i|_{Y_0}$ are local systems on $Y_{\geq 0} \setminus Y_0$ and Y_0 , respectively. Moreover, $K_i \in D_{\Delta}^b(\mathbb{C}_X)$. Then, we can easily deduce that $K_i = 0$. \blacksquare

10.4 Complex analyticity

10.4.1 Support

Let $K \in \mathbf{E}_{\odot}^b(IC_X)$. Let $\mathcal{S}(K)$ denote the collection of closed subsets C in X such that $K|_{X \setminus C} = 0$. Let $Z = \bigcap_{C \in \mathcal{S}(K)} C$.

Lemma 10.15 *We have $K = 0$ in $\mathbf{E}_{\mathbb{R}\text{-}c}^b(IC_{(X \setminus Z, X)})$.*

Proof We obtain the claim by using the curve test and an argument in the proof of Lemma 3.19. \blacksquare

10.4.2 Complex analyticity of the support

Let Z_k^{sm} denote the set of k -dimensional smooth points of Z . By an argument in the proof of Lemma 3.19, we can show that we have a subanalytic closed subset $(Z_k^{\text{sm}})^{(1)} \subset Z_k^{\text{sm}}$ with $\dim_{\mathbb{R}}(Z_k^{\text{sm}})^{(1)} \leq k - 1$ such that the following holds.

- Let \mathcal{C} be any connected component of $Z_k^{\text{sm}} \setminus (Z_k^{\text{sm}})^{(1)}$. Then, we have a tuple of subanalytic functions $\Lambda(\mathcal{C})$ on (\mathcal{C}, X) such that $\pi^{-1}(\mathbb{C}_{\mathcal{C}}) \otimes K = \bigoplus_{g_i \in \Lambda(\mathcal{C})} \mathbb{C}_X^{\mathbb{E}} \otimes^+ \mathbb{C}_{t \geq g_i}$.

In particular, $K|_{\mathcal{C}}$ comes from a local system on \mathcal{C} .

Lemma 10.16 *Z_k^{sm} are complex analytic submanifold of X . In particular, Z_k^{sm} are empty unless k is even.*

Proof Suppose that Z_k^{sm} is not a complex submanifold. We have a point $P \in Z_k^{\text{sm}} \setminus (Z_k^{\text{sm}})^{(1)}$ such that Z_k^{sm} is not a complex manifold at P . Let \mathcal{C} be the connected component of $Z_k^{\text{sm}} \setminus (Z_k^{\text{sm}})^{(1)}$ which contains P . Then, we have a holomorphic map $\varphi : \Delta \rightarrow X$ such that $\varphi(0) = P$ and that $\varphi^{-1}(\mathcal{C})$ is real one dimensional. Then, the support of $E\varphi^{-1}K$ is real one dimensional, which contradicts with the condition $K \in \mathbf{E}_{\Delta}^b(X)$. Hence, we obtain that Z_k^{sm} is a complex submanifold of X . \blacksquare

Lemma 10.17 *The closure $\overline{Z_k^{\text{sm}}}$ of Z_k^{sm} in X is a complex analytic subset in X .*

Proof We set $Y_k := \overline{Z_k^{\text{sm}}}$. Let Y_k^{sm} denote the set of smooth points, and we set $\text{Sing}(Y_k) := Y_k \setminus Y_k^{\text{sm}}$. By Proposition 10.6, it is enough to prove that $\dim_{\mathbb{R}} \text{Sing}(Y_k) \leq k - 2$. We assume that $\dim_{\mathbb{R}} \text{Sing}(Y_k) = k - 1$, and we shall deduce a contradiction.

Let us consider the case $k = \dim_{\mathbb{R}} Z$. By Proposition 10.7, we have a subanalytic closed subset $W \subset \text{Sing}(Y_k)$ with $\dim W \leq k - 2$ such that the following holds for any $P \in \text{Sing}(Y_k) \setminus W$.

- P is a smooth point of $\text{Sing}(Y_k)$.
- P is not contained in $Y_{k'}$ for any $k' < k$. Note that $Y_{k-1} = \emptyset$.
- Take a small neighbourhood of X_P of P in X . Then, we have a closed complex submanifold $\tilde{Y}_{k,P}$ and a real analytic function $\tilde{Y}_{k,P} \rightarrow \mathbb{R}$ whose exterior derivative is nowhere vanishing, such that $Y_k \cap X_P = \tilde{Y}_{k,P} \cap f^{-1}(\mathbb{R}_{\geq 0})$.

Then, we can take a holomorphic $\varphi : \Delta \rightarrow X$ such that $\varphi^{-1}(Z_k^{\text{sm}})$ and $\varphi^{-1}(X \setminus Z)$ are non-empty open subsets of Δ . It contradicts with $E\varphi^{-1}(K)$ comes from a cohomologically holonomic \mathcal{D}_Δ -complex. Hence, we are done in the case $k = \dim_{\mathbb{R}} Z$.

Suppose that we have already known that $Y_{k'}$ are complex subvarieties for any $k > k'$. By using Corollary 10.10, it is enough to prove that $Y_k \setminus \bigcup_{k' > k} Y_{k'}$ is a complex subvariety of $X \setminus \bigcup_{k' > k} Y_{k'}$, which can be argued as in the case of $k = \dim_{\mathbb{R}} Z$. Thus, the proof of Lemma 10.17 is completed. \blacksquare

Lemma 10.18 Z_k^{sm} is a complex analytic subset of X .

Proof Note that $Z = \bigcup_{k \geq 0} \overline{Z_k^{\text{sm}}}$. We have $\overline{Z_k^{\text{sm}}} \setminus Z_k^{\text{sm}} = \text{Sing}(\overline{Z_k^{\text{sm}}}) \cup \bigcup_{j \neq k} (\overline{Z_j^{\text{sm}}} \cap \overline{Z_k^{\text{sm}}})$, which is closed and complex analytic. Hence, we obtain the claim of the lemma. \blacksquare

10.4.3 Complex analyticity of singular locus

Lemma 10.19 We have a closed subset $(Z_k^{\text{sm}})_0^{(1)} \subset (Z_k^{\text{sm}})^{(1)}$, which is subanalytic in X , with the following property.

- $\dim_{\mathbb{R}}(Z_k^{\text{sm}})_0^{(1)} \leq k - 2$,
- For any point $P \in (Z_k^{\text{sm}})^{(1)} \setminus (Z_k^{\text{sm}})_0^{(1)}$, we take a connected component \mathcal{C} of $Z_k^{\text{sm}} \setminus (Z_k^{\text{sm}})^{(1)}$ such that P is contained in the closure of \mathcal{C} . Then, the functions $g \in \Lambda(\mathcal{C})$ are bounded around P .
- $\pi^{-1}(\mathbb{C}_{(Z_k^{\text{sm}})^{(1)})} \otimes K$ is controlled by bounded functions around any point $P \in (Z_k^{\text{sm}})^{(1)} \setminus (Z_k^{\text{sm}})_0^{(1)}$.

Proof We have a closed subanalytic subset $(Z_k^{\text{sm}})_1^{(1)} \subset Z_k^{\text{sm}}$ with $\dim(Z_k^{\text{sm}})_1^{(1)} \leq k - 2$ such that $\pi^{-1}(\mathbb{C}_{(Z_k^{\text{sm}})^{(1)})} \otimes K$ is controlled by bounded functions around any point $P \in (Z_k^{\text{sm}})^{(1)} \setminus (Z_k^{\text{sm}})_1^{(1)}$.

Let \mathcal{C} be a connected component of $Z_k^{\text{sm}} \setminus (Z_k^{\text{sm}})^{(1)}$. By Lemma 2.2, we have $Z_2(\mathcal{C}) \subset \partial\mathcal{C}$ such that (i) $\dim_{\mathbb{R}} Z_2(\mathcal{C}) \leq k - 2$, (ii) $Z_2(\mathcal{C})$ contains the singular locus of $\partial\mathcal{C}$, (iii) around any $P \in \partial\mathcal{C} \setminus Z_2(\mathcal{C})$, g_i are ramified analytic.

Take any $P \in (Z_k^{\text{sm}})^{(1)} \setminus \left(\bigcup_{\mathcal{C}} Z_2(\mathcal{C}) \cup (Z_k^{\text{sm}})_1^{(1)} \right)$. We take a connected component \mathcal{C} of $Z_k^{\text{sm}} \setminus (Z_k^{\text{sm}})^{(1)}$ such that the closure of \mathcal{C} contains P . Suppose that $g_{i_0} \in \Lambda(\mathcal{C})$ is not bounded around P . It is ramified analytic at P . Then, we have a neighbourhood U of P in $(Z_k^{\text{sm}})_0^{(1)}$ such that g_{i_0} is not bounded around any $Q \in U$. We can find a holomorphic map $\varphi : \Delta \rightarrow X$ such that (i) $\varphi(0) = P$, (ii) $\varphi(\Delta) \subset Z_k^{\text{sm}}$, (iii) $\varphi^{-1}((Z_k^{\text{sm}})^{(1)})$ is real 1-dimensional. The function $\varphi^{-1}g_{i_0}$ is unbounded around any point of $\varphi^{-1}((Z_k^{\text{sm}})^{(1)})$. However, it contradicts with the curve test. Hence, we obtain the claim of Lemma 10.19. \blacksquare

Lemma 10.20 The restriction of K to $Z_k^{\text{sm}} \setminus (Z_k^{\text{sm}})_0^{(1)}$ comes from a direct sum of local systems with shifted degrees.

Proof Take a neighborhood U_P of P in Z_k^{sm} . We set $W_P := U_P \cap (Z_k^{\text{sm}})^{(1)}$, and $U_P^\circ := U_P \setminus W_P$. We may assume that W_P and the connected components of U_P° are simply connected. By construction, the restrictions of K to U_P° and W_P comes from direct sums of shifts of local systems. By [3, Proposition 4.7.15], we can conclude that $K|_{U_P}$ comes from a cohomologically constructible complex on U_P . We may assume that we have $U_P = U_{P,0} \times \Delta$ and $W_P = U_{P,0} \times \{w \in \Delta \mid \text{Im}(w) = 0\}$. By the curve test, we obtain that $K|_{U_{P,0} \times \{w\}}$ comes from a direct sum of shifts of local systems. Then, the claim of the lemma follows. \blacksquare

By Lemma 10.20, we have a closed subset $A \subset Z_k^{\text{sm}}$ such that (i) A is subanalytic in X (ii) $\dim_{\mathbb{R}} A \leq k-2$, (iii) $\mathcal{H}^i(K|_{Z_k^{\text{sm}} \setminus A})$ come from local systems on $Z_k^{\text{sm}} \setminus A$. We assume that A is the minimum among such closed subanalytic subsets. Let A_m^{sm} denote the set of m -dimensional smooth points of A .

Proposition 10.21 A_m^{sm} is a complex submanifold of Z_k^{sm} . In particular, $A_m^{\text{sm}} = \emptyset$ unless m is even.

Proof We set $Y := Z_k^{\text{sm}}$ and $H := A_m^{\text{sm}}$ to simplify the notation. We take any point P of H . We assume that $T_P H$ is not a \mathbb{C} -subspace of $T_P Y$, and we shall derive a contradiction. By shrinking Y , we may assume that $T_P H$ are not \mathbb{C} -subspaces of $T_P Y$ for any $P \in H$.

Let us consider the case where $k-m=2$. Let P be any point of H . We have the real subspace $T_P H \subset T_P Y$. We have a complex line $\mathcal{V} \subset T_P Y$ such that $\mathcal{V} \cap T_P H = \{0\}$ by Lemma 10.3. We take a holomorphic map $\Phi : \Delta_z \times \Delta_w^{m/2} \rightarrow Y$ such that (i) $T_{(z,w)} \Phi : T_{(z,w)} \Delta_z \times \Delta_w \rightarrow T_{\Phi(z,w)} Y$ are isomorphisms for any (z, w) , (ii) $T_{(0,0)} \Phi(T_0 \Delta_z) = \mathcal{V}$. By shrinking Y , we may assume that Φ is an isomorphism, by which we identify Y and $\Delta_z \times \Delta_w^{m/2}$. We may also assume to have a \mathbb{C} -valued real analytic function $h : \Delta_w^{m/2} \rightarrow \Delta$ such that $\Phi^{-1}(H) \subset \Delta_z \times \Delta_w^{m/2}$ is the graph of h . We set $\zeta(z, w) := z - h(w)$. The restriction of ζ to $\Delta_z \times \{w\}$ is holomorphic. We have the open embedding $Y \rightarrow \mathbb{C}_{\zeta} \times \mathbb{C}_w^{n-1}$ given by (ζ, w) .

Let $\varpi : \tilde{Y}(H) \rightarrow Y$ be the oriented real blowing up along H . Let $\zeta = re^{\sqrt{-1}\theta}$ be the polar decomposition. Then, (r, θ, w) gives a local coordinate around Q for any $Q \in \partial \tilde{Y}(H)$.

We have a closed subanalytic subset $\tilde{Y}(H)^{(1)} \subset \tilde{Y}(H)$ with $\dim_{\mathbb{R}}(\tilde{Y}(H)^{(1)}) = \dim_{\mathbb{R}} Y - 1$ such that (i) $\partial \tilde{Y}(H) \subset \tilde{Y}(H)^{(1)}$, (ii) we have subanalytic functions g_1, \dots, g_p on $(\tilde{Y}(H) \setminus \tilde{Y}(H)^{(1)}, \tilde{Y}(H))$, and integers m_i , such that

$$\pi^{-1}(\mathbb{C}_{\tilde{Y}(H) \setminus \tilde{Y}(H)^{(1)}}) \otimes E\varpi^{-1}(K) \simeq \bigoplus_{i=1}^p \mathbb{C}^E \otimes \mathbb{C}_{t \geq g_i}[m_i].$$

Let W denote the closure of $\tilde{Y}(H)^{(1)} \setminus \partial \tilde{Y}(H)$. We have a closed subanalytic subset $R \subset \partial \tilde{Y}(H)$ and $Z \subset H$ such that the following holds.

- $\dim_{\mathbb{R}} R < \dim_{\mathbb{R}} \partial \tilde{Y}(H)$ and $\dim_{\mathbb{R}} Z < \dim_{\mathbb{R}}(H)$.
- R contains $W \cap \partial \tilde{Y}(H)$. Moreover, g_1, \dots, g_p are ramified analytic around any $Q \in \partial \tilde{Y}(H) \setminus R$.
- $R \setminus \varpi^{-1}(Z) \rightarrow H$ is horizontal.

By enlarging $\tilde{Y}(H)^{(1)}$ and Z , we may assume that $R \setminus \varpi^{-1}(Z)$ is contained in W .

We take $P_1 = (z_1, w_1) \in H \setminus Z$ and $Q_1 \in \varpi^{-1}(P_1) \setminus R$. Let us prove that any g_i are bounded around Q_1 . Suppose that some g_{i_0} is unbounded around Q_1 , and we shall derive a contradiction. By moving Q_1 , we may assume that $g_{i_0} = r^{-e/\rho} g_{i_0}^{(0)}$, where $g_{i_0}^{(0)}$ is nowhere vanishing analytic function of $(r^{1/\rho}, \theta, w)$, and e and ρ are positive integers.

Note that $\varpi^{-1}(P_1)$ is identified with $(T_{P_1} Y / T_{P_1} H \setminus \{0\}) / \mathbb{R}_{>0}$. We take $u \in T_{P_1} Y / T_{P_1} H \setminus \{0\}$ such that $[u] = Q_1$. Because it is assumed that $T_{P_1} H$ is not a \mathbb{C} -subspace, we have $v \in T_{P_1} H$ such that $\sqrt{-1}v$ is mapped to u by the projection $T_{P_1} Y \rightarrow T_{P_1} Y / T_{P_1} H$ by Lemma 10.4. We take an analytic map $\gamma :]-\epsilon_1, \epsilon_1[\rightarrow H$ such that $\gamma(0) = P_1$ and $\gamma'(0) = v$. Let $\varphi : \Delta_{\epsilon_2} \rightarrow Y$ be the complexification of γ . Let $x + \sqrt{-1}y$ be the real coordinate on Δ_{ϵ_2} . We have $T_{(0,0)} \varphi(\partial_x) = \gamma'(0) = v$ and $T_{(0,0)} \varphi(\partial_y) = \sqrt{-1}v$. Hence, $\varphi^{-1}(H) = \Delta_{\epsilon_2} \cap \mathbb{R}$. Moreover, we have $\varphi(\{(x, y) \in \Delta_{\epsilon_2} \mid y > 0\}) \subset \tilde{Y}(H) \setminus \tilde{Y}(H)^{(1)}$, and $\varphi^*(g_{i_0})$ is unbounded around any point of $\Delta_{\epsilon_2} \cap \mathbb{R}$. It contradicts with the assumption $E\varphi^{-1}K$ comes from a holonomic \mathcal{D} -module. Hence, we obtain that g_i are bounded around $Q_1 \in \varpi^{-1}(P_1) \setminus R$.

Then, we can derive the following lemma easily.

Lemma 10.22 For any point $P' \in H \setminus Z$, let $\varphi_{P'} : \Delta_{\epsilon} \rightarrow Y$ be the holomorphic map given by $\varphi_{P'}(a) = (a + h(P'), P')$. Then, $E\varphi_{P'}^{-1}K$ comes from a regular holonomic \mathcal{D} -module. \blacksquare

We may regard $\tilde{Y}(H)$ as an open subset of $\{0 \leq r\} \times S^1 \times H$. Let $\eta : \{0 \leq r\} \times S^1 \times H \rightarrow \{0 \leq r\} \times H$ be the projection. We have a closed subanalytic subset $B \subset \{0 \leq r\} \times H$ with the following property.

- We have $\dim_{\mathbb{R}} B = \dim_{\mathbb{R}} H$, and B contains $\{0\} \times H$.

- Each connected component of $(\{0 \leq r\} \times H) \setminus B$ is simply connected.
- $\tilde{Y}(H)^{(1)} \setminus \eta^{-1}(B) \longrightarrow \{0 \leq r\} \times H$ is proper and a local diffeomorphism.
- We have subanalytic functions $g_i^{(1)}$ on $\tilde{Y}(H)^{(1)} \setminus \eta^{-1}(B)$, and integers $m_i^{(1)}$, such that

$$\pi^{-1}(\mathbb{C}_{\tilde{Y}(H)^{(1)} \setminus \eta^{-1}(B)}) \otimes K \simeq \bigoplus_{i=1}^m \mathbb{C}^E \otimes \mathbb{C}_{t \geq g_i^{(1)}}[m_i^{(1)}].$$

On each connected component \mathcal{C} of $(\{0 \leq r\} \times H) \setminus B$, the set $\eta^{-1}(\mathcal{C}) \cap \tilde{Y}(H)^{(1)}$ is described as the union of subanalytic functions $h_i^{\mathcal{C}} : \mathcal{C} \longrightarrow S^1$. Set $B_1 := B \setminus (\{0\} \times H)$. We set $Z_1 := Z \cup (\overline{B_1} \cap (\{0\} \times H))$.

By Lemma 10.22, the restriction of g_i to $(\Delta_z \times \{P_2\}) \cap (\tilde{Y}(H) \setminus \tilde{Y}(H)^{(1)})$ are bounded for any $P_2 \in \Delta_w^{n-1} \setminus Z_1$. We also have the boundedness of the restrictions of $g_i^{(1)}$ to $(\Delta_z \times \{P_2\}) \cap (\tilde{Y}(H)^{(1)} \setminus \partial \tilde{Y}(H))$ are bounded for any $P_2 \in \Delta_w^{n-1} \setminus Z_1$. Then, by Lemma 2.21, for any $P_2 \in \Delta_w^{n-1} \setminus Z_1$, we have a neighbourhood U of P_2 in Δ_w^{n-1} such that the restrictions of g_i to $(\Delta_z \times U) \cap (\tilde{Y}(H) \setminus \tilde{Y}(H)^{(1)})$ are bounded, and that the restrictions of $g_i^{(1)}$ to $(\Delta_z \times U) \cap (\tilde{Y}(H)^{(1)} \setminus \partial \tilde{Y}(H))$ are bounded.

Then, we have closed subanalytic subset $Z_2 \subset H$ with $\dim_{\mathbb{R}} Z_2 < \dim H$ such that the following holds.

- For any $P \in H \setminus Z_2$, we have a neighbourhood Y_P of P in Y such that $K|_{Y_P}$ comes from an object in $D_{\mathbb{R}-c}^b(Y_P)$.

Then, H has to be a complex submanifold by Lemma 10.11, and we have arrived at a contradiction in the case $k - m = 2$.

Let us consider the case $k - m > 2$. Let $\varphi : \Delta \longrightarrow Y$ be any holomorphic map such that $\varphi(0) \in H$, and $\varphi(\Delta) \not\subset H$. Because $\dim_{\mathbb{R}} H < \dim_{\mathbb{R}} Y - 2$, we have a holomorphic map $\Phi : \Delta \times \Delta \longrightarrow Y$ such that (i) $\Phi|_{\Delta \times \{0\}} = \varphi$, (ii) $\dim_{\mathbb{R}} \Phi^{-1}(H) \leq 1$. Note that $\Phi^{-1}(H)$ is real analytic. Then, we have a complex curve $\mathcal{Z} \subset \Delta \times \Delta$ which contains $\Phi^{-1}(H)$. Let $j : \mathcal{Y} = (\Delta^2 \setminus \mathcal{Z}, \Delta^2) \longrightarrow \Delta^2$ be the inclusion. We have $\mathbf{E}j_!! \mathbf{E}j^{-1} \mathbf{E}\Phi^{-1}K \in \mathbf{E}_{\odot}(IC_{\mathcal{Y}})$. It comes from a meromorphic flat bundle \mathcal{V} on (Δ^2, \mathcal{Z}) . By considering the restriction to a neighbourhood of any point of $\mathcal{Z} \setminus \Phi^{-1}(H)$, we obtain that \mathcal{V} is regular singular. Hence, $\mathbf{E}\varphi^{-1}K$ comes from a regular singular holonomic \mathcal{D} -complex on Δ .

We take a closed subanalytic subset $Z_{10} \subset H$ such that (i) $\dim_{\mathbb{R}} Z_{10} < \dim_{\mathbb{R}} H$, (ii) $K|_{H \setminus Z_{10}}$ comes from a local system. Then, we obtain that $K|_{Y \setminus Z_{10}}$ comes from a \mathbb{R} -constructible complex on Y . Then, we obtain that $H \setminus Z_{10}$ has to be a complex submanifold by Lemma 10.13, and we arrived at a contradiction. Thus, the proof of Proposition 10.21 is completed. \blacksquare

Lemma 10.23 *The closure \overline{A} of A in X is a complex analytic subvariety of X .*

Proof Let $\overline{A_m^{\text{sm}}}$ denote the closure of A_m^{sm} in X . It is enough to prove that $\overline{A_m^{\text{sm}}}$ are complex analytic subvarieties of X . By Corollary 10.10, it is enough to prove that $A'_m := \overline{A_m^{\text{sm}}} \cap Z_k^{\text{sm}}$ is a complex subvariety in Z_k^{sm} . Let $\text{Sing}(A'_m)$ denote the singular locus of A'_m . We have only to prove that $\dim_{\mathbb{R}} \text{Sing} A'_m < m - 1$.

Let us consider the case $m = \dim_{\mathbb{R}} A$. We assume that $\dim_{\mathbb{R}} \text{Sing} A'_m = m - 1$, and we shall deduce a contradiction. We have closed subanalytic subset $W'_m \subset \text{Sing} A'_m$ with $\dim_{\mathbb{R}} W'_m < m - 1$ such that the following holds for any $P \in \text{Sing} A'_m \setminus W'_m$:

- P is a smooth point of $\text{Sing} A'_m$.
- Take a small neighbourhood X_P of P in X . Then, we have a complex submanifold $\tilde{A}'_{m,P}$ of X_P and a real analytic function $f_P : \tilde{A}'_{m,P} \longrightarrow \mathbb{R}$ whose exterior derivative is nowhere vanishing, such that $A'_m = \tilde{A}'_{m,P} \cap f_P^{-1}(\mathbb{R}_{\geq 0})$.

We set $Z_{k,P} := Z_k \cap X_P$. We take any complex hypersurface $H_P \subset X_P$ such that $Z_{k,P} \not\subset H_P$ and $\tilde{A}'_{m,P} \subset H_P$. By our construction, the set $(Z_{k,P} \cap H_P) \setminus A'_m$ is non-empty. Let $j_{H_P} : (X_P \setminus H_P, X_P) \longrightarrow (X_P, X_P)$ be the inclusion of the bordered spaces. By Theorem 9.3, each cohomology of $\mathbf{E}j_{H_P}!! \mathbf{E}j_{H_P}^{-1}K|_{X_P}$ comes from a

meromorphic flat bundle (V, ∇) on $(Z_{k,P}, H_P \cap Z_{k,P})$. The restriction of (V, ∇) to any point of $(H_P \cap Z_{k,P}) \setminus A'_m$ is regular singular. Hence, we obtain that (V, ∇) is regular singular. By varying H_P , we obtain that $K|_{X_P}$ comes from an object of $D_{\mathbb{R}\text{-c}}^b(\mathbb{C}_{X_P})$.

After enlarging W'_m , we may assume that each cohomology K_P^i of $K|_{X_P}$ satisfies that (i) $K_P^i|_{Z_{k,P} \setminus A'_m}$ is a local system on $Z_{k,P} \setminus A'_m$, (ii) $K_P^i|_{X_P \cap (A'_m \setminus \text{Sing}(A'_m))}$ is a local system on $X_P \cap (A'_m \setminus \text{Sing}(A'_m))$, (iii) $K_P^i|_{X_P \cap \text{Sing}(A'_m)}$ is a local system on $X_P \cap \text{Sing}(A'_m)$. By Lemma 10.14, we obtain that $K|_{X_P}$ are local systems on X_P , which contradicts with our choice of A . Hence, we obtain $\dim_{\mathbb{R}} \text{Sing } A'_m < m - 1$, and hence A'_m is a complex analytic subvariety in the case $m = \dim_{\mathbb{R}} A$.

Assume that we have already known that $A'_{m'}$ are complex analytic subvarieties for $m' > m$. By Corollary 10.10, it is enough to prove that $A'_m \setminus \bigcup_{m' > m} A'_{m'}$ is a complex analytic subvariety of $X \setminus \bigcup_{m' > m} A'_{m'}$, which can be argued as in the case of $m = \dim A$. Thus, we obtain Lemma 10.23. \blacksquare

Let P be any point of Z . We have a small neighbourhood X_P of P and the irreducible decomposition $X_P \cap Z = \bigcup Z_i$ as a germ of complex analytic sets at P . For each Z_i , we can take a complex hypersurface H_i of X_P such that (i) $Z_i \not\subset H_i$, (ii) $\bigcup_{j \neq i} Z_j \subset H_i$, (iii) $Z_i \setminus H_i$ is smooth, (iv) $K|_{Z_i \setminus H_i}$ comes from an \mathbb{R} -constructible complex whose cohomology sheaves are local systems. Let $\varphi_i : \tilde{Z}_i \rightarrow Z_i$ be a projective morphism such that (i) \tilde{Z}_i is a complex manifold, (ii) $\tilde{H}_i := \varphi_i^{-1}(H_i)$ is normal crossing, (ii) $\tilde{Z}_i \setminus \tilde{H}_i \simeq Z_i \setminus H_i$. Let $\iota_i : Z_i \rightarrow X$ be the inclusion. Let $j_i : (Z_i \setminus H_i, Z_i) \rightarrow Z_i$ be the inclusion of the bordered spaces. We have $\text{E}j_{i!!} \text{E}(\iota_i \circ \varphi_i)^{-1} K$ in $\text{E}_{\Delta}^b(IC_{\tilde{Z}_i})$.

Lemma 10.24 *We have a cohomologically holonomic $\mathcal{D}_{\tilde{Z}_i}$ -complex V with an isomorphism $\text{DR}^{\text{E}} V \simeq \text{E}j_{i!!} \text{E}(\iota_i \circ \varphi_i)^{-1} K$.*

Proof By the construction, the k -th cohomology of $\text{E}j_{i!!} \text{E}(\iota_i \circ \varphi_i)^{-1} K$ come from a meromorphic flat bundles with a shift of degree. By the fully faithfulness of the Riemann-Hilbert correspondence [3], we obtain that $\text{E}j_{i!!} \text{E}(\iota_i \circ \varphi_i)^{-1} K$ comes from a cohomologically holonomic $\mathcal{D}_{\tilde{Z}_i}$ -complex. \blacksquare

Let $k_i : (X_P \setminus H_i, X_P) \rightarrow X_P$ be the inclusion of the bordered spaces. We obtain the following from the previous lemma.

Lemma 10.25 *$\text{E}k_{i!!} \text{E}k_i^{-1} K$ comes from a cohomologically holonomic \mathcal{D}_{X_P} -complexes.* \blacksquare

10.5 Proof of Theorem 10.1

Let us prove Theorem 10.1. It is enough to prove the essential surjectivity. For that purpose, it is enough to prove the following for any $K \in \text{E}_{\Delta}^b(IC_X)$.

- $\Upsilon^{\text{E}}(K)$ are objects in $D_{\text{hol}}^b(\mathcal{D}_X)$.
- The natural morphisms $K \rightarrow \text{Sol}^{\text{E}} \Upsilon^{\text{E}}(K)$ are isomorphisms.

We have only to consider these properties locally around any point of X . We use an induction on the dimension of the support of K .

Let Z be the support of K . Let $P \in Z$. Let X_P denote a small neighbourhood of P in X . Let Z_1 denote the union of the $\dim(Z)$ -dimensional components of Z . We have a hypersurface H_P of X_P such that (i) $Z_1 \not\subset H_P$, (ii) $Z_1 \setminus H_P$ is smooth, (iii) $K|_{Z_1 \setminus H_P}$ comes from an \mathbb{R} -constructible complex whose cohomology sheaves are local systems. Let $j : (X_P \setminus H_P, X_P) \rightarrow X_P$ be the inclusion of the bordered spaces. As remarked, $K_1 := \text{E}j_{!!} \text{E}j^{-1} K$ comes from a cohomologically holonomic \mathcal{D}_{X_P} -complex. Hence, we have a cohomologically holonomic \mathcal{D}_{X_P} -complex \mathcal{M} with an isomorphism $\text{Sol}^{\text{E}} \mathcal{M} \simeq K_1$. We have $j_* \mathcal{M} \simeq \mathcal{M}$. According to [3], we have a canonical isomorphism $\Upsilon^{\text{E}} \text{Sol}^{\text{E}}(\mathcal{M}) \simeq \mathcal{M}$. We have the following morphisms:

$$\text{Sol}^{\text{E}}(\mathcal{M}) \xrightarrow{a_1} \text{Sol}^{\text{E}} \Upsilon^{\text{E}} \text{Sol}^{\text{E}}(\mathcal{M}) \xrightarrow[\simeq]{a_2} \text{Sol}^{\text{E}}(\mathcal{M})$$

Here, a_1 is $\Phi_{\text{Sol}^{\text{E}}(\mathcal{M})}$, and a_2 is the isomorphism induced by $\Upsilon^{\text{E}} \text{Sol}^{\text{E}}(\mathcal{M}) \simeq \mathcal{M}$. It is easy to check that the restriction of $a_2 \circ a_1$ to $Y \setminus H$ is an isomorphism, and hence that $a_2 \circ a_1$ is an isomorphism. Hence, we obtain that $\Upsilon^{\text{E}}(K_1)$ is a cohomologically holonomic \mathcal{D}_{X_P} -complexes, and $K_1 \rightarrow \text{Sol}^{\text{E}} \Upsilon^{\text{E}}(K_1)$ is an isomorphism.

We have the natural morphism $K_1 \rightarrow K|_{X_P}$ in $E_\Delta^b(X|_P)$. We have the distinguished triangle $K_1 \rightarrow K|_{X_P} \rightarrow K_2 \rightarrow K_1[1]$ in $E_\Delta^b(X|_P)$. We can apply the assumption of the induction to K_2 . Then, we obtain that $\Upsilon^E K|_{X_P}$ comes from a cohomologically holonomic \mathcal{D}_{X_P} -complex, and the natural morphism $K|_{X_P} \rightarrow \text{Sol}^E \Upsilon^E(K|_{X_P})$ is an isomorphism. ■

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Address

Research Institute for Mathematical Sciences, Kyoto University, Kyoto 606-8502, Japan
takuro@kurims.kyoto-u.ac.jp